Binomial Theorem

1 The x-independent term in the expansion of $(3-ax)^{2n+1}$ is 81 times the x-independent term in the expansion of $(3+x)^n$. Find n.

Observe that the 1st term of both binomials are the x-independent terms

$\therefore \frac{3^{2n+1}}{3^n} = 81$

$3^{n+1} = 81$

$3^n = 27$

$3^n = 3^3$

$n = 3$

2 In the expansion of $\left(ax+\frac{b}{3x}\right)^n$,the fifth term is 1120 and the coefficient of the x^2 term is 1792, where a, b and n are positive constants.

a) Write down the value of n

b) Find the values of a and b

a) $T_{r+1} = \binom{n}{r}(ax)^{n-r}\left(\frac{b}{3x}\right)^r$

$T_5 = \binom{n}{4}(ax)^{n-4}\left(\frac{b}{3x}\right)^4$

Equating x-terms: $x^{n-4}x^{-4} = x^0$ (fifth term is x-independent)

$n-4-4=0$

$n=8$

b) $T_5 = \binom{8}{4}(ax)^{8-4}\left(\frac{b}{3x}\right)^4$

$1120 = 70(a^4x^4)(\frac{b^4}{3^4x^4})$

$a^4b^4 = 1296$

$ab = 6 \quad -(1)$

x^2 term: $T_{r+1} = \binom{8}{r}(ax)^{8-r}\left(\frac{b}{3x}\right)^8$

Find r of x^2 term: $x^{8-\mathrm{r}}x^{-\mathrm{r}} = x^2$

$8-2r = 2$

$r = 3$

$T_4 = \binom{8}{3}(ax)^{8-3}\left(\frac{b}{3x}\right)^3$

$1792 = 56(a^5x^5)\left(\frac{b^3}{3^3x^3}\right)$

$a^5b^3 = 864$

$a^2(ab)^3 = 864 \quad -(2)$

Sub (1) into (2):

$a^2(6)^3 = 864$

$a^2 = 4$

$a = 2\ or -2\ (rej)$

$b = 3$

3 Expand, in ascending power of x, up to the x^2 term for the following binomials:

$$(5+x)^5$$

$$\left(2-\frac{x}{5}\right)^5$$

Hence, find the coefficient of the x^2 term in the expansion of $\left(10+x-\frac{x^2}{5}\right)^5$

Applying Binomial Expansion Formula and simplify:

$$(5+x)^5 = 3125 + 3125x + 1250x^2 + \cdots$$

$$\left(2-\frac{x}{5}\right)^5 = 32 - 16x + \frac{16x^2}{5} + \cdots$$

Since $(5+x)\left(2-\frac{x}{5}\right) = (10 + x - \frac{x^2}{5})$,

$\left(10+x-\frac{x^2}{5}\right)^5 = (5+x)^5\left(2-\frac{x}{5}\right)^5$

$= (3125 + 3125x + 1250x^2 + \cdots)(32 - 16x + \frac{16x^2}{5} + \cdots)$

x^2-coefficient $= 3125\left(\frac{16}{5}\right) + 3125(-16) + 1250(32)$

$= 0$

4 Expand $(1+2x-2x^2)^8$ in ascending powers of x up until the x^3 term. Using a suitable numerical value of x, find an approximate value of 1.001998^8, giving your answer correct to 4 significant figures.

$(1+2x-2x^2)^8 = 1 + \binom{8}{1}(2x-2x^2) + \binom{8}{2}(2x-2x^2)^2 + \binom{8}{3}(2x-2x^2)^3 + \cdots$

$= 1 + 16x - 16x^2 + 28(4x^2 + 4x^4 - 8x^3) + 56[(4x^2 + 4x^4 - 8x^3)(2x - 2x^2)] + \cdots$

$= 1 + 16x - 16x^2 + 28(4x^2 - 8x^3 + \cdots) + 56[8x^3 + \cdots]$

$= 1 + 16x - 96x^2 + 224x^3 + \cdots$

$1 + 2x - 2x^2 = 1.001998$

$2x^2 - 2x + 0.001998 = 0$

$x = 0.001$ or $x = 0.999$

$\therefore 1.0198^8 = (1 + 2(0.001) - 2(0.001)^2)^8$

$= 1 + 16(0.001) - 96(0.001)^2 + 224(0.001)^3 + \cdots$

≈ 1.016 (4s.f.)

5 In the binomial expansion of $(3+x)^n$, where n is a positive integer, the coefficients of x^3 and x^4 are in the ratio $\frac{3}{2}$. Find the value of n.

$$(3+x)^n = 3^n + \binom{n}{1}3^{n-1}x + \binom{n}{2}3^{n-2}x^2 + \binom{n}{3}3^{n-3}x^3 + \binom{n}{4}3^{n-4}x^4 + \cdots$$
$$= 3^n + n3^{n-1}x + \frac{n(n-1)}{2}3^{n-2}x^2 + \frac{n(n-1)(n-2)}{1\times2\times3}3^{n-3}x^3 + \frac{n(n-1)(n-2)(n-3)}{1\times2\times3\times4}3^{n-4}x^4 + \cdots$$
$$\frac{\frac{n(n-1)(n-2)}{1\times2\times3}3^{n-3}}{\frac{n(n-1)(n-2)(n-3)}{1\times2\times3\times4}3^{n-4}} = \frac{3}{2}$$
$$\frac{\frac{\cancel{n(n-1)(n-2)}}{\cancel{1\times2\times3}}3^{n-3}}{\frac{\cancel{n(n-1)(n-2)}(n-3)}{\cancel{1\times2\times3}\times4}3^{n-4}} = \frac{3}{2}$$
$$\frac{4\times3^n3^{-3}}{(n-3)3^n3^{-4}} = \frac{3}{2}$$
$$\frac{4\times3}{n-3} = \frac{3}{2}$$
$$24 = 3n - 9$$
$$n = 11$$

6 The first four terms in the expansion of $(1+ax)^n$ is $1+18x+15a^2x^2+bx^3+\cdots$
Find the value of n, a and b.

$(1+ax)^n = 1+\binom{n}{1}ax+\binom{n}{2}a^2x^2+\binom{n}{3}a^3x^3+\cdots$
$= 1+nax+\frac{n(n-1)}{1\times 2}a^2\,x^2+\frac{n(n-1)(n-2)}{1\times 2\times 3}a^3x^3+\cdots$

$1+18x+15a^2x^2+bx^3+\cdots = 1+nax+\frac{n(n-1)}{1\times 2}a^2\,x^2+\frac{n(n-1)(n-2)}{1\times 2\times 3}a^3x^3+\cdots$
Comparing coefficients:
$18 = na$ -(1)
$15a^2 = \frac{n(n-1)}{2}a^2$
$15 = \frac{n(n-1)}{2}$
$n^2-n-30=0$
$(n+5)(n-6)=0$
$n=-5$ (Rej) or $n=6$

Substitute $n=6$ into (1):
$18=6a$
$a=3$

$b=\frac{n(n-1)(n-2)}{1\times 2\times 3}a^3$
$b=540$

7 In the expansion of $(2+5x)^n$, the first three terms in ascending power of x are denoted by $a, b\ and\ c$ respectively. Given that $\frac{ac}{b^2}=\frac{3}{7}$, find the value of n.

$$(2+5x)^n = 2^n + \binom{n}{1}2^{n-1}5x + \binom{n}{2}2^{n-2}25x^2 + \cdots$$
$$= 2^n + n(2^{n-1})5x + \frac{n(n-1)}{2}(2^{n-2})25x^2 + \cdots$$

$a = 2^n$
$b = n(2^{n-1})5x$
$c = \frac{n(n-1)}{2}(2^{n-2})25x^2$

$$\frac{ac}{b^2} = \frac{(2^n)\frac{n(n-1)}{2}(2^{n-2})25x^2}{(n(2^{n-1})5x)^2}$$
$$= \frac{n(n-1)(2^{2n-3})25x^2}{n^2(2^{2n-2})\,25x^2}$$
$$= \frac{n-1}{2n}$$
$$\frac{3}{7} = \frac{n-1}{2n}$$
$6n = 7n - 7$
$n = 7$

8 Find the value of q if the coefficient of x in the expansion of $(1+2x)^5\left(1-\frac{1}{3}x\right)^6-(1+x)^9(1+qx)^4$ is zero.
State the term independent of x in the above equation.

$(1+2x)^5\left(1-\frac{1}{3}x\right)^6-(1+x)^9(1+qx)^4$
$=(1+10x+\cdots)(1-2x+\cdots)-(1+9x+\cdots)(1+4qx+\cdots)$

Coefficient of $x = 10\times 1-2\times 1-4q\times 1-9\times 1$
Given that coefficient of $x=0$,
$0=10-2-4q-9$
$q=0.25$

Value of x −independent term $=1\times 1-1\times 1=0$

9 In the expansion of $\left(\frac{x}{3}+\frac{3}{x}\right)^n$ in descending powers of x, the ratio of the coefficient of the third term to that of the fourth term is $\frac{1}{15}$. Find the value of n.

$$\left(\frac{x}{3}+\frac{3}{x}\right)^n = \left(\frac{x}{3}\right)^n + n\left(\frac{x}{3}\right)^{n-1}\left(\frac{3}{x}\right) + \frac{n(n-1)}{2}\left(\frac{x}{3}\right)^{n-2}\left(\frac{3}{x}\right)^2 + \frac{n(n-1)(n-2)}{6}\left(\frac{x}{3}\right)^{n-3}\left(\frac{3}{x}\right)^3 + \cdots$$
$$= \left(\frac{x}{3}\right)^n + n\left(\frac{x}{3}\right)^{n-2} + \frac{n(n-1)}{2}\left(\frac{x}{3}\right)^{n-4} + \frac{n(n-1)(n-2)}{6}\left(\frac{x}{3}\right)^{n-6} + \cdots$$

Third term coefficient $= \frac{n(n-1)}{2}\left(\frac{1}{3}\right)^{n-4}$

Fourth term coefficient $= \frac{n(n-1)(n-2)}{6}\left(\frac{1}{3}\right)^{n-6}$

$$\frac{coefficient\ of\ 3rd\ term}{coefficient\ of\ 4th\ term} = \frac{1}{15}$$
$$\frac{\frac{n(n-1)}{2}\left(\frac{1}{3}\right)^{n-4}}{\frac{n(n-1)(n-2)}{6}\left(\frac{1}{3}\right)^{n-6}} = \frac{1}{15}$$
$$15\,\frac{n(n-1)}{2}\left(\frac{1}{3}\right)^{n-4} = \frac{n(n-1)(n-2)}{6}\left(\frac{1}{3}\right)^{n-6}$$
$$15 = \frac{n-2}{3}\left(\frac{1}{3}\right)^{-2}$$
$$15 = 3n - 6$$
$$n = 7$$

10 Given the expression $\left(\frac{2x\sqrt{x}-1}{2\sqrt{x}}\right)^n$ where the 11th term of the expansion, in descending powers of x is independent of x. Find the value of this term.

$\left(\frac{2x\sqrt{x}-1}{2\sqrt{x}}\right)^n = \left(x - \frac{1}{2\sqrt{x}}\right)^n$
$T_{11} = \binom{n}{10}(x)^{n-10}\left(-\frac{1}{2}x^{-0.5}\right)^{10}$
$= \binom{n}{10}(1)^{n-10}\left(-\frac{1}{2}\right)^{10}(x^{n-10-5})$
Given that T_{11} is the x −independent term,
$x^{n-10-5} = x^0$
$n = 15$
$T_{11} = \binom{15}{10}(x)^{15-10}\left(-\frac{1}{2}x^{-0.5}\right)^{10}$
$= 2.93$ (3 s.f.)