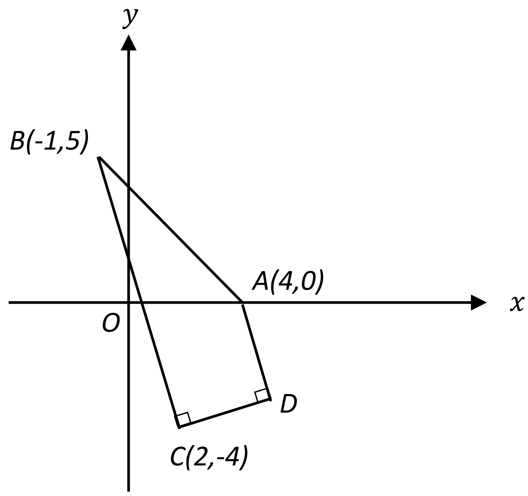
Coordinate Geometry



1 The diagram shows a trapezium ABCD such that BC is parallel to AD and perpendicular to CD.
i) Find the coordinates of vertex D
ii) Point E lies on BC such that the area of triangle ACE is $\frac{1}{2}$ of the area of triangle ABE. Find the coordinates of E.
iii) Point F lies on AD produce such that it forms a parallelogram with vertices A, B and C. Find the possible coordinates of F.
iv) Determine the ratio of the area of triangle ACB to the parallelogram AFBC.

i) $Gradient\ of\ AD = Gradient\ of\ BC = \frac{(5-(-4))}{(-1-2)} = -3$

$Gradient\ of\ CD = \frac{-1}{-3} = \frac{1}{3}$

$Equation\ of\ AD: (y-0) = -3(x-4)$

$y = -3x + 12 \quad ---(1)$

$Equation\ of\ CD: \left(y-(-4)\right) = \frac{1}{3}(x-2)$

$y = \frac{1}{3}x - \frac{14}{3}$

$3y = x - 14 \quad ---(2)$

$Sub\ (1)\ into\ (2):$

$3(-3x+12) = x - 14$

$-9x + 36 - x + 14 = 0$

$x = 5$

$y = -3$

ii) $\frac{area\ of\ ACE}{area\ of\ ABE} = \frac{1}{2}$

$\frac{\frac{1}{2} \times EC \times Height}{\frac{1}{2} \times EB \times Height} = \frac{1}{2}$

$\frac{EC}{EB} = \frac{1}{2}$

By Ratio Theorem, Coordinates of E $= \left(\frac{1\times -1+2\times 2}{2+1}, \frac{1\times 5+2\times -4}{2+1}\right) = (-1,-1)$

iii) Case 1: For parallelogram arranged as AFBC

$\overrightarrow{BC} = \overrightarrow{FA} = \begin{pmatrix} 3 \\ -9 \end{pmatrix}$

$\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AF} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix}$

$Coordinates\ of\ F\ is\ (1,9)$

Case 2: For parallelogram arranged as ABFC

$\overrightarrow{BC} = \overrightarrow{AF} = \begin{pmatrix} 3 \\ -9 \end{pmatrix}$

$\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AF} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ -9 \end{pmatrix} = \begin{pmatrix} 7 \\ -9 \end{pmatrix}$

$Coordinates\ of\ F\ is\ (7,-9)$

$\therefore possible\ coordinates\ of\ F\ are\ (1,9) and\ (7,-9)$

iv) $\frac{area\ of\ ACB}{area\ of\ AFBC} = \frac{\frac{1}{2} \times BC \times height}{BC \times height} = \frac{1}{2}$

2 Point A has coordinates $(2,3)$ and line l_1 has equation $2y = 4x + 5$.

a) Find the coordinates of the foot of the perpendicular from Point A to line l_1.

b) Find the shortest distance from Point A to line l_1

c) Point B is the reflection of Point A on the line l_1, find the coordinates of B.

a) $Let\ the\ foot\ of\ the\ perpendicular\ from\ Point\ A\ to\ line\ l_1\ be\ Point\ C$

$Gradient\ of\ l_1 = \frac{4}{2} = 2$

$Gradient\ of\ \perp\ line = -\frac{1}{2}$

$Equation\ of\ \perp\ line: (y-3) = -\frac{1}{2}(x-2)$

$y = -\frac{1}{2}x + 4$

$Equating\ l_1\ and\ \perp\ line: 2x + \frac{5}{2} = -\frac{1}{2}x + 4$

$4x + 5 + x - 8 = 0$

$5x = 3$

$x = 0.6$

$y = 3.7$

$Coordinates\ of\ C\ is\ (0.6, 3.7)$

b) Length of $AC = \sqrt{(2-0.6)^2 + (3-3.7)^2}$

$= 1.57\ units\ (3\ s.f.)$

c) C is the mid-point of AB

$\frac{2+x}{2} = 0.6$

$x = -0.8$

$\frac{3+y}{2} = 3.7$

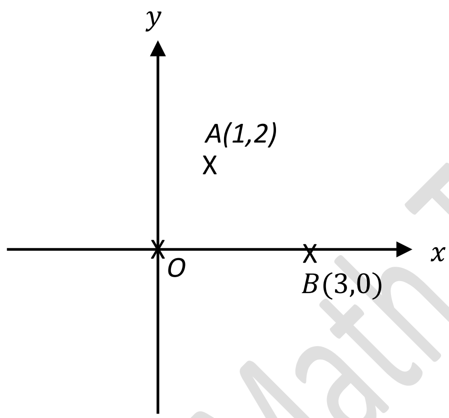
$y = 4.4$

Coordinates of B is $(-0.8, 4.4)$

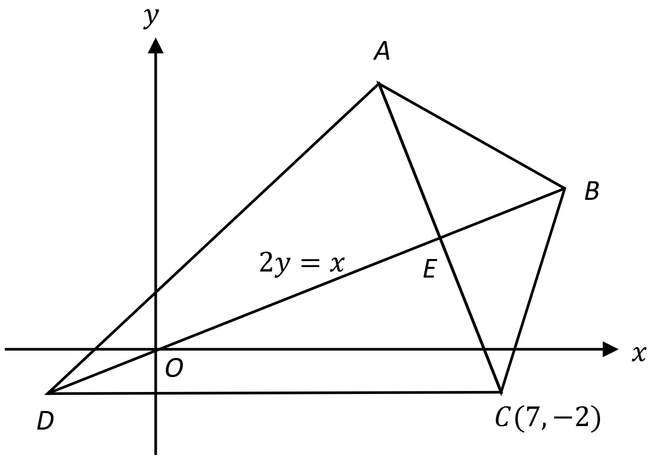
3 The equation of the perpendicular bisector of the line segment which joins $A(2,3)$ and $B(h,k)$ is $y = x - 1$. Find the value of h and of k.

$Gradient\ of\ perpendicular\ bisector = 1$
$Gradient\ of\ AB\ = -1 = \frac{(k-3)}{(h-2)}$
$k - 3 = -h + 2$
$k = -h + 5 \quad ---(1)$
$Midpoint\ of\ AB = \left(\frac{2+h}{2}, \frac{3+k}{2}\right)$
$\left(\frac{3+k}{2}\right) = \left(\frac{2+h}{2}\right) - 1$
$3 + k = 2 + h - 2$
$k = h - 3 \quad ---(2)$
$Sub\ (1)\ to\ (2): -h + 5 = h - 3$
$2h = 8$
$h = 4$
$k = 1$

4 The diagram shows 3 vertices of a parallelogram. Given A(1,2), B(3,0) and O, find the possible positions of the fourth vertex.



Ans: $(4,2), (-2,2), (2,-2)$



5 The diagram above (not drawn to scale) shows kite $ABCD$ with DC parallel to the x-axis. The area of triangle ADC is 3 times that of triangle ABC. Given that $C(7,-2)$ and the equation of the diagonal BD is $2y = x$, find

i) Coordinates of D

ii) Coordinates of E

iii) Coordinates of A

iv) Coordinates of B

i) y-coordinate of $D = -2$

Sub $y = -2$ into $2y = x$:

$x = -4$

$Coordinates\ of\ D\ is\ (-2,-4)$

ii) $Gradient\ of\ DB = \frac{1}{2}$

$Gradient\ of\ AC = -2$

$Equation\ of\ AC: \left(y-(-2)\right) = -2(x-7)$

$y = -2x + 12$

$Equate\ AC\ and\ BD: -2x + 12 = \frac{1}{2}x$

$24 = 5x$

$x = 4.8$

$y = 2.4$

$Coordinates\ of\ E\ is\ (4.8, 2.4)$

iii) E is the midpoint of AC.

$\left(\frac{x+7}{2}, \frac{y-2}{2}\right) = (4.8, 2.4)$

$x = 2.6$

$y = 6.8$

$Coordinates\ of\ A\ is\ (2.6, 6.8)$

iv) Given that $\frac{Area\ ADC}{Area\ ABC} = \frac{3}{1}$

$\frac{DE}{EB} = \frac{3}{1}$

$Using\ Ratio\ Theorem: \left(\frac{1\times(-2)+3\times(x)}{3+1}, \frac{1\times(-4)+3\times(y)}{3+1}\right) = (4.8, 2.4)$

$x = 7.07$

$y = 4.53$

$Coordinates\ of\ B\ is\ (7.07, 4.53)$

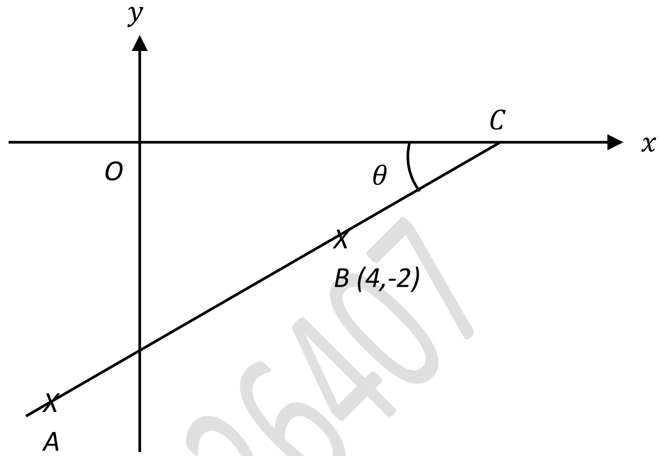
6 Three points A, B and C lies on a straight line such that $AB = 2BC$. The coordinates of point B is $(4, -2)$ and $\tan\theta = \frac{2}{3}$. Find the

i) equation of line AC

ii) coordinates of A and C

iii) coordinates of the point on line AC that is closest to O.

(Leave you answer to the nearest 3 s.f.)



i) Since $\tan\theta = \frac{2}{3}$,

Gradient of $AC = \frac{2}{3}$

Equation of AC: $\left(y - (-2)\right) = \frac{2}{3}(x - 4)$

$y = \frac{2}{3}x - \frac{8}{3} - 2$

$3y = 2x - 14$ -(1)

ii) C is on the x-axis, Sub $y = 0$,

$0 = 2x - 14$

$x = 7$

$\therefore \boldsymbol{C(7, 0)}$

$AB = 2BC$

$\sqrt{(x-4)^2 + (y+2)^2} = 2\sqrt{(4-7)^2 + (-2-0)^2}$

$(x-4)^2 + (y+2)^2 = 4(9+4)$

$(x-4)^2 + (y+2)^2 = 52$ -(2)

Sub (1) into (2):

$(x-4)^2 + \left(\frac{2x-14}{3} + 2\right)^2 = 52$

$(x-4)^2 + \left(\frac{2}{3}x - \frac{8}{3}\right)^2 = 52$

$x^2 + 16 - 8x + \frac{4}{9}x^2 + \frac{64}{9} - \frac{32}{9}x - 52 = 0$

$13x^2 - 104x - 260 = 0$

$(x+2)(x-10) = 0$

$x = -2$ or $x = 10\ (Rej)$

$y = -6$

$\therefore \boldsymbol{A(-2, -6)}$

iii) Let the point on AC that is closest to O be D

Gradient of $OD = \frac{-1}{\frac{2}{3}} = -\frac{3}{2}$

Equation of OD: $y = -\frac{3}{2}x$ -(3)

Sub (1) with (3):

$3\left(-\frac{3}{2}x\right) = 2x - 14$

$-9x = 4x - 28$

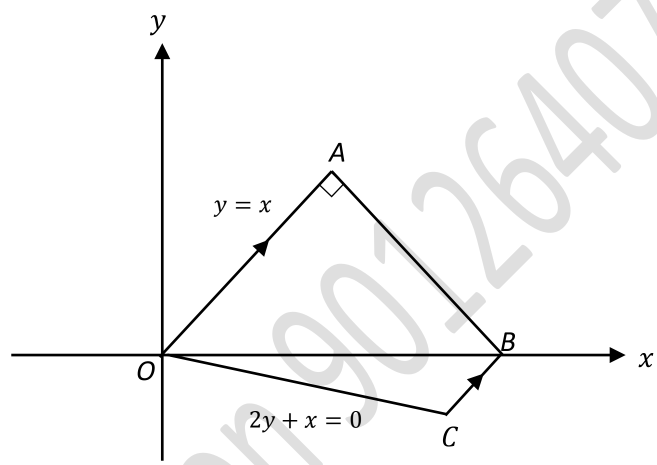
$x = \frac{28}{13} = 2.15\ (3\ s.f.)$

$y = -\frac{42}{13} = -3.23\ (3\ s.f.)$

Coordinates of point is $(\boldsymbol{2.15, -3.23})$

7 The diagram shows a trapezium OABC. The equation of OA is $y = x$ and the equation of OC is $2y + x = 0$. Line OA is parallel to CB and perpendicular to AB. Point B is on the x-axis. The length of OA is $4\sqrt{2}$ units.

i) Find the coordinates of A
ii) Find the coordinates of B
iii) Find the coordinates of C.
iv) Hence, calculate the area of trapezium $OABC$.



i) Let the coordinates of A be (x, x)
$OA = \sqrt{(x-0)^2 + (x-0)^2}$
$4\sqrt{2} = \sqrt{2x^2}$
$32 = 2x^2$
$x = 4$ or $x = -4\ (Rej)$
$y = 4$
Coordinate of A is $(4,4)$

ii) Gradient of OA $= 1$
Gradient of AB $= -1$
Equation of AB: $(y-4) = -1(x-4)$
$y = -x + 8$
When $y = 0$: $0 = -x + 8$
$x = 8$
Coordinates of B is $(8,0)$

iii) Gradient of CB $=$ Gradient of OA $= 1$
Equation of CB: $(y-0) = 1(x-8)$
$y = x - 8$
Equate equation OC with equation CB:
$-\frac{x}{2} = x - 8$
$-x = 2x - 16$
$x = \frac{16}{3} = 5\frac{1}{3}$
$y = -\frac{8}{3} = -2\frac{2}{3}$
Coordinate of C is $(5\frac{1}{3}, -2\frac{2}{3})$

iv) $Area\ OABC = \frac{1}{2}\begin{vmatrix} 4 & 0 & 5\frac{1}{3} & 8 & 4 \\ 4 & 0 & -2\frac{2}{3} & 0 & 4 \end{vmatrix}$
$= \frac{1}{2}\left(\left(4\times 0 + 0\times -2\frac{2}{3} + 5\frac{1}{3}\times 0 + 8\times 4\right) - \left(4\times 0 + 0\times 5\frac{1}{3} - 2\frac{2}{3}\times 8 + 0\times 4\right)\right)$
$= \frac{1}{2}\left(32 + \frac{64}{3}\right)$
$= 26\frac{2}{3}\ units^2$

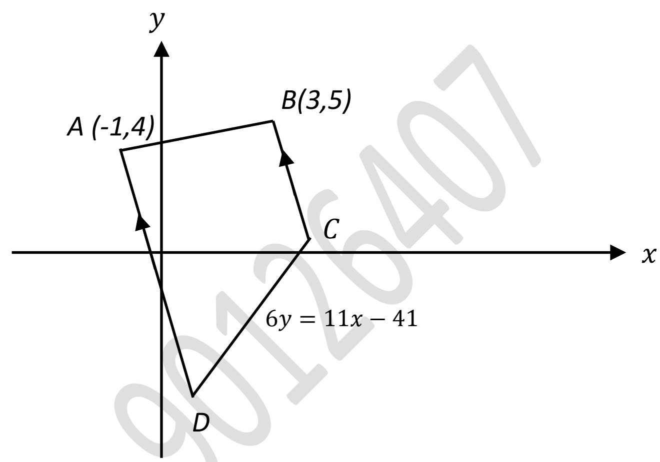
8 ABCD is a trapezium with AB parallel to BC. The equation of DC is $6y = 11x - 41$. Given that midpoint of AD lies on the y-axis and the midpoint of BD lies on the x-axis, find

i) the coordinates of D

ii) the coordinates of C

iii) area of ABCD

iv) the perpendicular distance between AD and BC (leaving your answer to 3 s.f.)



i) Let the coordinates of D be (x, y)

Since the midpoint of AD lies on y-axis,

$\frac{x-1}{2} = 0$

$x = 1$

Since the midpoint of BD lies on the x-axis,

$\frac{y+5}{2} = 0$

$y = -5$

The coordinates of D is $(1, -5)$

ii) Gradient of $AD = \frac{4-(-5)}{-1-1} = -\frac{9}{2}$

Gradient of BC = Gradient of AD = $-\frac{9}{2}$

Equation of BC: $(y - 5) = -\frac{9}{2}(x - 3)$

$y = -\frac{9}{2}x + 18.5$

Equate BC and DC:

$-\frac{9}{2}x + 18.5 = \frac{11}{6}x - \frac{41}{6}$

$x = 4$

$y = 0.5$

The coordinates of C is $(4, 0.5)$

iii) Area of ABCD = $\frac{1}{2}\left|\begin{array}{ccccc} -1 & 1 & 4 & 3 & -1 \\ 4 & -5 & 0.5 & 5 & 4 \end{array}\right|$

$= \frac{1}{2}\left((-1 \times -5 + 1 \times 0.5 + 4 \times 5 + 3 \times 4) - (4 \times 1 - 5 \times 4 + 0.5 \times 3 + 5 \times -1)\right)$

$= \frac{1}{2}(57)$

$= 28.5\ units^2$

iv) Length of AD = $\sqrt{(4-(-5))^2 + ((-1)-1)^2} = \sqrt{85}$

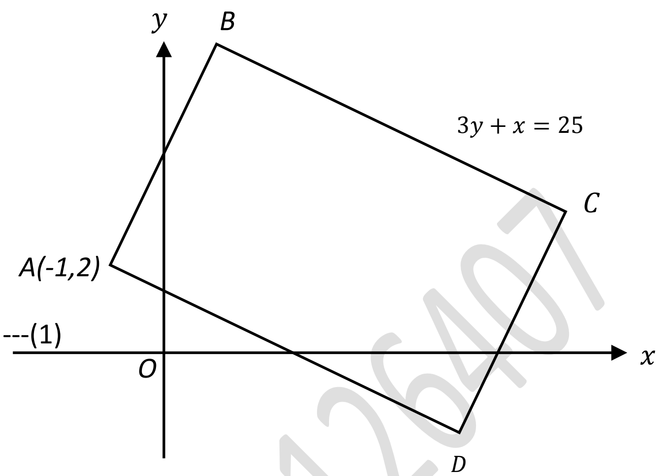
Length of BC = $\sqrt{(5-0.5)^2 + (4-3)^2} = \sqrt{21.25}$

Area of ABCD = $\frac{AD+BC}{2} \times Height$

$28.5 = \frac{\sqrt{85}+\sqrt{21.25}}{2} \times Height$

$Height = 4.12\ units$

9 In the diagram, ABCD is a rectangle. The coordinates of A are (-1,2) and the equation of BC is $3y + x = 25$. Given that the area of ABCD is $80\ units^2$, find the coordinates of B, C and D.



Equation of BC: $y = -\frac{x}{3} + \frac{25}{3}$ ---(1)

Gradient of BC $= -\frac{1}{3}$

Gradient of AB $= 3$

Equation of AB: $(y - 2) = 3(x + 1)$

$y = 3x + 5$

Equate equations of AB and BC:

$3x + 5 = -\frac{x}{3} + \frac{25}{3}$

$x = 1$

$y = 8$

Coordinates of B are $(1,8)$

Length of AB $= \sqrt{(8-2)^2 + \left(1-(-1)\right)^2} = \sqrt{40}$

Length of BC $= \frac{80}{\sqrt{40}} = 4\sqrt{10}$

Let the coordinates of C be (x, y)

Length of BC $= \sqrt{(8-y)^2 + (1-x)^2} = 4\sqrt{10}$

$(8-y)^2 + (1-x)^2 = 160$ ---(2)

Substitute Equation (1) with (2):

$\left(8 + \frac{x}{3} - \frac{25}{3}\right)^2 + (1-x)^2 = 160$

Solve for x:

$x = 13$ or $x = -11\ (Rej)$

$y = 4$

Coordinates of C are $(13, 4)$

Equation of CD: $(y - 4) = 3(x - 13)$

$y = 3x - 35$

Equation of AD: $(y - 2) = -\frac{1}{3}(x + 1)$

$y = -\frac{1}{3}x + \frac{5}{3}$

Equate Equations of CD with AD:

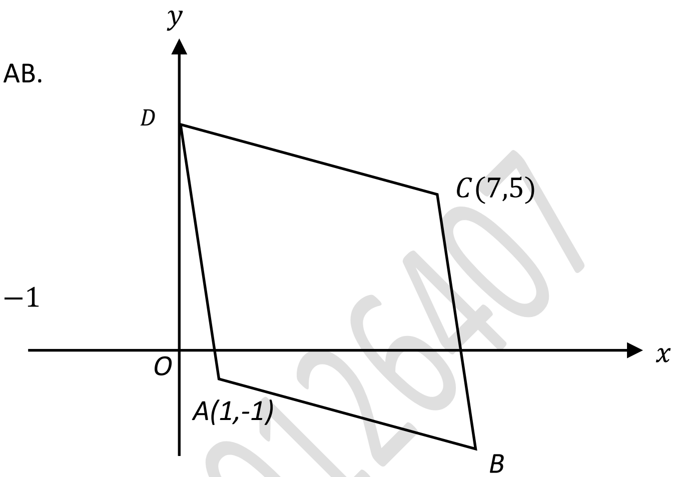
$3x - 35 = -\frac{1}{3}x + \frac{5}{3}$

$x = 11$

$y = -3$

Coordinates of D are $(11, -3)$

10 The diagram shows a rhombus ABCD. Two of the points are A(1,-1) and C(7,5). Point D lies on the y-axis.
i) Find the coordinates of D
ii) Find the coordinates of B
iii) Find the area of rhombus ABCD
iv) Calculate the perpendicular distance from C to AB.



i) Mid point of AC $= \left(\frac{7+1}{2}, \frac{5-1}{2}\right) = (4,2)$
Gradient of AC$= \frac{(5-(-1))}{(7-1)} = 1$
Gradient of perpendicular bisector of AC $= -\frac{1}{1} = -1$
Equation of DB: $(y-2) = -1(x-4)$
$y = -x + 6$
When $x = 0, y = 0 + 6 = 6$
Coordinates of D are $(0,6)$
ii) Let the coordinates of B be (x, y),
Mid point of DB = Mid point of AC
$\left(\frac{0+x}{2}, \frac{6+y}{2}\right) = (4,2)$
$\frac{0+x}{2} = 4$ and $\frac{6+y}{2} = 2$
$x = 8$ and $y = -2$
Coordinates of B are $(8, -2)$
iii) Area of ABCD $= \frac{1}{2}\begin{vmatrix} 1 & 8 & 7 & 0 & 1 \\ -1 & -2 & 5 & 6 & -1 \end{vmatrix}$
$= \frac{1}{2}\left((1 \times -2 + 8 \times 5 + 7 \times 6 + 0 \times -1) - (-1 \times 8 - 2 \times 7 + 5 \times 0 + 6 \times 1)\right)$
$= \frac{1}{2}(89)$
$= 44.5\ units^2$
iv) Length of AB $= \sqrt{(-2-(-1))^2 + (8-1)^2} = \sqrt{50}$
Perpendicular distance $= \frac{44.5}{\sqrt{50}} = 6.29\ units\ (3\ s.f.)$