Differentiation

1 Sand is poured onto a surface at a rate of $3\pi\ cm^3 s^{-1}$ and forms a right circular cone. The height of the cone is always 3 times the radius. Find the rate of change of the radius 9 seconds after the pouring started.

$h = 3r$ ---(1)
$V = \frac{1}{3}\pi r^2 h$ ---(2)

Sub Equation (1) with (2):
$V = \frac{1}{3}\pi r^2 (3r)$
$V = \pi r^3$ ---(3)
$\frac{dV}{dr} = 3\pi r^2$

After 9 Seconds, volume of sand $= 9 \times 3\pi = 27\pi\ cm^3$
Sub $V = 27\pi$ into Equation (3):
$27\pi = \pi r^3$
$r = 3cm$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$
$\frac{dr}{dt} = \frac{dV}{dt} \div \frac{dV}{dr}$
$= \frac{3\pi}{3\pi(3)^2}$
$= \frac{1}{9}\ cm\ s^{-1}$

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2 Differentiate $y=\sqrt[3]{\frac{(3x^2+3)^2}{(2x^2+2x+1)}}$ with respect to x.

$$y=\left(\frac{(3x^2+3)^2}{(2x^2+2x+1)}\right)^{\frac{1}{3}}$$

$$y=\frac{(3x^2+3)^{\frac{2}{3}}}{(2x^2+2x+1)^{\frac{1}{3}}}$$

$$\frac{dy}{dx}=\frac{\left(\frac{2}{3}\right)(3x^2+3)^{-\frac{1}{3}}(6x)(2x^2+2x+1)^{\frac{1}{3}}-(3x^2+3)^{\frac{2}{3}}\left(\frac{1}{3}\right)(2x^2+2x+1)^{-\frac{2}{3}}(4x+2)}{(2x^2+2x+1)^{\frac{2}{3}}}$$

$$=\frac{(4x)(2x^2+2x+1)-(3x^2+3)\left(\frac{4}{3}x+\frac{2}{3}\right)}{(2x^2+2x+1)^{\frac{2}{3}}(3x^2+3)^{\frac{1}{3}}(2x^2+2x+1)^{\frac{2}{3}}}$$

$$=\frac{8x^3+8x^2+4x-4x^3-4x-2x^2-2}{(2x^2+2x+1)^{\frac{4}{3}}(3x^2+3)^{\frac{1}{3}}}$$

$$=\frac{4x^3+6x^2-2}{(2x^2+2x+1)^{\frac{4}{3}}(3x^2+3)^{\frac{1}{3}}}$$

$$=\frac{2(2x^3+3x^2-1)}{(2x^2+2x+1)^{\frac{4}{3}}(3x^2+3)^{\frac{1}{3}}}$$

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3 The normal to the curve $y = 3x^2 + kx + 2$ at the point $(-2,4)$ is parallel to the line $7y + x = 14$. Find the value of k and calculate the coordinates of the point where this normal meets the curve again, giving your answers corrected to 3 s.f.

$7y + x = 14$
$y = -\frac{1}{7}x + 2$
Gradient $= -\frac{1}{7}$

$y = 3x^2 + kx + 2$
$\frac{dy}{dx} = 6x + k$
Gradient of tangent $= 6(-2) + k = -12 + k$
Gradient of Normal $= \frac{1}{k-12}$
$-\frac{1}{7} = \frac{1}{k-12}$
$k = 5$
Equation of Normal: $(y - 4) = -\frac{1}{7}(x + 2)$
$y = -\frac{1}{7}x + \frac{26}{7}$
Equate Normal with Curve:
$-\frac{1}{7}x + \frac{26}{7} = 3x^2 + 5x + 2$
$-x + 26 = 21x^2 + 35x + 14$
$21x^2 + 36x - 12 = 0$
$7x^2 + 12x - 4 = 0$
$(7x - 2)(x + 2) = 0$
$x = 0.286$ (3 s.f.) or -2
$y = 3.67$ (3 s.f.) or 4
Coordinates is $(0.286, 3.67)$.

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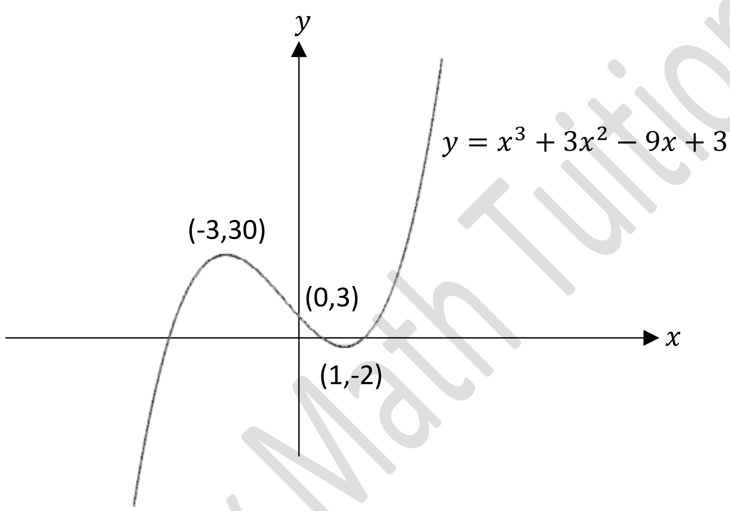
4 Sketch the curve $y = x^3 + 3x^2 - 9x + 3$, clearly show the y intercept and all the turning points.

$\frac{dy}{dx} = 3x^2 + 6x - 9$
$\frac{dy}{dx} = 0$
$0 = 3x^2 + 6x - 9$
$(x + 3)(x - 1) = 0$
$x = -3$ or $x = 1$
$y = 30$ or $y = -2$

	1^-	1	1^+	-3^-	-3	-3^+
$\frac{dy}{dx}$	< 0	0	> 0	> 0	0	< 0
Slope of tangent	\	–	/	/	–	\

$\therefore (-3, 30)$ is a maximum point and $(1, -2)$ is a minimum point.
$Sub\ x = 0, y = 3$



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5 A 20 cm piece of wire is cut into 2 pieces. One piece is bent to form a circle while the other piece is bent to form a square. Find the minimum area enclosed by the two pieces.

Let the piece of wire used to form the circle be x cm and the piece of wire used to form a square be $(20-x)$ cm

Circumference of Circle $= 2\pi r$

$x = 2\pi r$

$r = \frac{x}{2\pi}$

Total area enclosed by both pieces, $A = l^2 + \pi r^2$

$A = \left(\frac{20-x}{4}\right)^2 + \pi\left(\frac{x}{2\pi}\right)^2$

$A = \frac{400+x^2-40x}{16} + \frac{x^2}{4\pi}$

$A = 25 + \left(\frac{1}{16} + \frac{1}{4\pi}\right)x^2 - \frac{5}{2}x$

$\frac{dA}{dx} = 2\left(\frac{1}{16} + \frac{1}{4\pi}\right)x - \frac{5}{2}$

$\frac{dA}{dx} = 0$

$0 = 2\left(\frac{1}{16} + \frac{1}{4\pi}\right)x - \frac{5}{2}$

$x = 8.798$

$A = 14\ cm^2$

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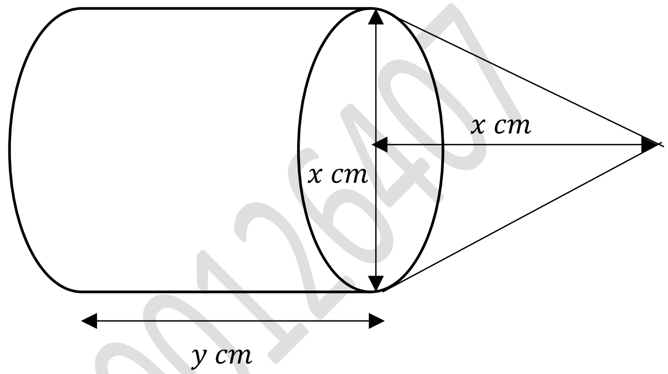
6 The diagram shows a solid consisting of a circular cone attached to a cylinder. The diameter of both the cylinder and cone is x cm, the length of the cylinder is y cm and the height of the cone is x cm. Given that the volume of the solid is $20\pi\ cm^3$,

a) Express y in terms of x

b) Show that the area of the school is given by $A = \frac{\pi x^2(3\sqrt{5}-5)}{12} + \frac{160\pi}{x}$.

c) Find the value of x for which A has a stationary value. Determine whether the corresponding value of A is a maximum or a minimum value.



a) $Volume = \pi\left(\frac{x}{2}\right)^2 y + \frac{1}{3}\pi\left(\frac{x}{2}\right)^2 x$

$20\pi = \frac{\pi x^2}{12}(3y + x)$

$3y = \frac{240}{x^2} - x$

$y = \frac{80}{x^2} - \frac{x}{3}$

b) $slanted\ height\ of\ cone, l = \sqrt{x^2 + \left(\frac{x}{2}\right)^2} = \sqrt{\frac{5x^2}{4}}$

$Area, A = \pi\left(\frac{x}{2}\right)^2 + 2\pi xy + \pi\left(\frac{x}{2}\right)l$

$= \pi\left(\frac{x}{2}\right)^2 + 2\pi x\left(\frac{80}{x^2} - \frac{x}{3}\right) + \pi\left(\frac{x}{2}\right)\sqrt{\frac{5x^2}{4}}$

$= \pi x\left(\frac{x}{4} + \frac{160}{x^2} - \frac{2x}{3} + \frac{x}{4}\sqrt{5}\right)$

$= \pi x^2\left(-\frac{5}{12} + \frac{3\sqrt{5}}{12}\right) + \frac{160\pi}{x}$

$= \frac{\pi x^2(3\sqrt{5}-5)}{12} + \frac{160\pi}{x}$ (Shown)

c) $\frac{dA}{dx} = \frac{\pi(3\sqrt{5}-5)}{6}x - \frac{160\pi}{x^2} = 0$

$\frac{\pi(3\sqrt{5}-5)}{6}x = \frac{160\pi}{x^2}$

$x = 8.252$

$\frac{d^2A}{dx^2} = \frac{\pi(3\sqrt{5}-5)}{6} + \frac{320\pi}{x^3}$

$= \frac{\pi(3\sqrt{5}-5)}{6} + \frac{320\pi}{(8.252)^3}$

$= 2.68 > 0$

The stationary point is a minimum value.

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7 Differentiate the following with respect to x.

a) $y = \ln(2x^2)^{\sin x}$

b) $y = \frac{\ln(\cos x)}{e^x}$

c) $y = \frac{\left(e^{4x^2+5}\right)\left(e^{3x-2x^2}\right)}{e^{2x^2+4}}$

a) $y = \ln(2x^2)^{\sin x}$

$y = \sin x \ln(2x^2)$

$\frac{dy}{dx} = \cos x \ln(2x^2) + \left(\frac{4x}{2x^2}\right) \sin x$

b) $y = \frac{\ln(\cos x)}{e^x}$

$\frac{dy}{dx} = \frac{\frac{-\sin x}{\cos x} e^x - \ln(\cos x) e^x}{e^{2x}}$

$= -\frac{\tan x + \ln(\cos x)}{e^x}$

c) $y = \frac{\left(e^{4x^2+5}\right)\left(e^{3x-2x^2}\right)}{e^{2x^2+4}}$

$y = e^{4x^2+5+3x-2x^2-2x^2-4}$

$y = e^{1+3x}$

$\frac{dy}{dx} = 3e^{1+3x}$

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8 The tangent to the curve $y = \frac{\ln x^2}{x^2}$ at the point where the curve crosses the positive x-axis and passes through the point $(2, k)$. Find the value of k.

When $y = 0$,

$0 = \frac{\ln x^2}{x^2}$

$\ln x^2 = 0$

$x^2 = 1$

$x = 1$ or $x = -1\ (Rej)$

$\frac{dy}{dx} = \frac{\frac{2x}{x^2}(x^2) - 2x \ln x^2}{x^4}$

$= \frac{2(1 - \ln x^2)}{x^3}$

Sub $x = 1$:

$\frac{dy}{dx} = \frac{2(1 - \ln 1^2)}{1} = 2$

Equation of tangent:

$(y - 0) = 2(x - 1)$

$y = 2x - 2$

Sub $x = 2$ into tangent equation:

$y = 2 \times 2 - 2 = 2$

$\therefore k = 2$

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9 Given that $2x + y = 12$, find the stationary value of $x^2 + y^2 + 5xy$ and determine the nature of this stationary point.

$2x + y = 12$

$y = 12 - 2x$ -(1)

Let $z = x^2 + y^2 + 5xy$ -(2)

Sub (1) into (2):

$z = x^2 + (12 - 2x)^2 + 5x(12 - 2x)$

$z = x^2 + 144 + 4x^2 - 48x + 60x - 10x^2$

$z = -5x^2 + 12x + 144$ -(3)

$\frac{dz}{dx} = -10x + 12$

At stationary point, $\frac{dz}{dx} = 0$

$0 = -10x + 12$

$x = \frac{6}{5}$

Sub $x = \frac{6}{5}$ into (3)

$z = -5\left(\frac{6}{5}\right)^2 + 12\left(\frac{6}{5}\right) + 144$

$z = 151.2$

$\frac{d^2z}{dx^2} = -10$

Since $\frac{d^2z}{dx^2} < 0$, the stationary point is a maximum point.

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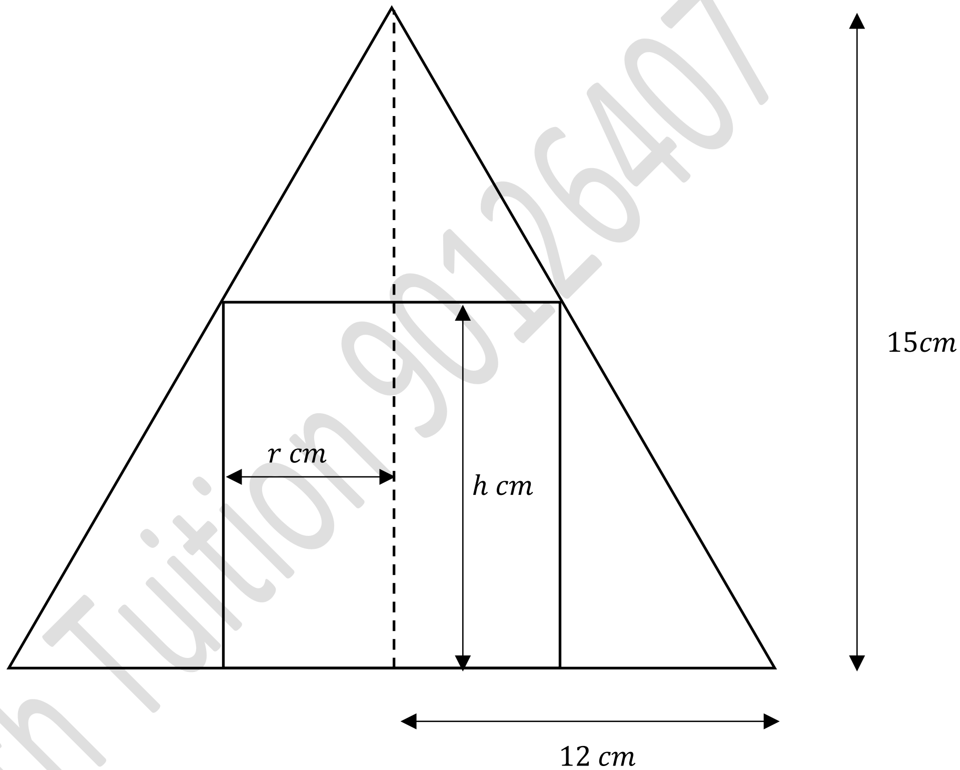
10 The diagram shows the cross-section of a hollow cone of height 15 cm and radius 12 cm. A solid cylinder of height h cm and radius r cm is placed inside the cone such that the upper circular edge of the cylinder is in contact with the inner wall of the cone.

a) Show that the volume of the cylinder is given by $V = 15\pi r^2 - \frac{5}{4}\pi r^3$

b) Given that r varies, find the value of r for which V has a stationary value.

c) Find the stationary value of V and determine its nature.



a) By similar triangles:

$\frac{15-h}{15} = \frac{r}{12}$

$180 - 12h = 15r$

$h = \frac{180-15r}{12}$

$V = \pi r^2 h$

$= \pi r^2 \left(\frac{180-15r}{12}\right)$

$= 15\pi r^2 - \frac{5}{4}\pi r^3$ (Shown)

b) $V = 15\pi r^2 - \frac{5}{4}\pi r^3$

$\frac{dV}{dr} = 30\pi r - \frac{15}{4}\pi r^2$

Stationary Point, $\frac{dV}{dr} = 0$

$30\pi r - \frac{15}{4}\pi r^2 = 0$

$15\pi r\left(2 - \frac{1}{4}r\right) = 0$

$r = 0$ (Rej) or $r = 8$

c) Sub $r = 8$,

$V = 15\pi(8)^2 - \frac{5}{4}\pi(8)^3$

$= 320\pi$

$= 1010\ cm^3\ (3\ s.f.)$

$\frac{dV}{dr} = 30\pi r - \frac{15}{4}\pi r^2$

$\frac{d^2y}{dx^2} = 30\pi - \frac{15}{2}\pi r$

When $r = 8$,

$\frac{d^2y}{dx^2} = 30\pi - \frac{15}{2}\pi(8)$

$= -30\pi < 0$

Since $\frac{d^2y}{dx^2} < 0$, stationary point is a maximum point.

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