Trigonometry

1 Find all the values of x between -3 and 2 for which $\sec\frac{\pi}{4}\ \sec^2(2x-2) = -\tan\frac{5\pi}{3}$

$\frac{2}{\sqrt{2}}\sec^2(2x-2) = \sqrt{3}$

$\sec^2(2x-2) = \frac{\sqrt{6}}{2}$

$\cos^2(2x-2) = \frac{2}{\sqrt{6}}$

$\cos(2x-2) = \pm 0.90360$ (All 4 Quadrants)

$\alpha = 0.4427$

$2x-2 = -8.982, -6.726, -5.840, -3.584, -2.699, -0.4427, 0.4427, 2.699$

$x = -3.49\ (Rej), -2.36, -1.92, -0.792, -0.350, 0.779, 2.35\ (Rej)$

2 Find all the angles between 0^o and 180^o for the equation $\cos 3x = -11\cos^2 x$.

$\cos 3x + 11\cos^2 x = 0$

$\cos(2x + x) + 11\cos^2 x = 0$

$\cos 2x \cos x - \sin 2x \sin x + 11\cos^2 x = 0$

$(2\cos^2 x - 1)\cos x - 2\sin x \cos x \sin x + 11\cos^2 x = 0$

$2\cos^3 x - \cos x - 2\sin^2 x \cos x + 11\cos^2 x = 0$

$2\cos^3 x - \cos x - 2(1-\cos^2 x)\cos x + 11\cos^2 x = 0$

$2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x + 11\cos^2 x = 0$

$4\cos^3 x + 11\cos^2 x - 3\cos x = 0$

$\cos x\,(4\cos^2 x + 11\cos x - 3) = 0$

$\cos x\,(4\cos x - 1)(\cos x + 3) = 0$

$\cos x = 0$ (All 4 Qs) or $\cos x = \frac{1}{4}$ (1st and 4th Q) or $\cos x = -3\ (Rej)$

$\alpha = 90^o$ or $\alpha = 75.5^o$

$x = 90^o, 270^o\ (Rej)$ or $x = 75.5^o, 284.5^o (Rej)$

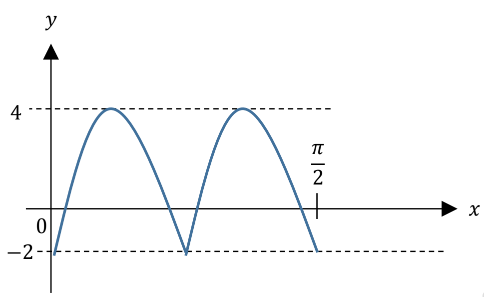
$x = 75.5^o, 90^o$

3 Solve the equation $\cos^2\theta + 3\sin\theta\cos\theta = -1$ for $0^o \leq \theta \leq 180^o$

$\cos^2\theta + 3\sin\theta\cos\theta = -1$
$\cos^2\theta + 3\sin\theta\cos\theta + \sin^2\theta + \cos^2\theta = 0$
$2\cos^2\theta + 3\sin\theta\cos\theta + \sin^2\theta = 0$
$(2\cos\theta + \sin\theta)(\cos\theta + \sin\theta) = 0$

$2\cos\theta = -\sin\theta$ or $\cos\theta = -\sin\theta$
$\tan\theta = -2$ (2nd and 4th Quadrants) or $\tan\theta = -1$ (2nd and 4th Quadrant)
$\alpha = 63.43^o$ or $\alpha = 45^o$
$\theta = 116.6^o, 243.4^o (Rej)$ or $\theta = 135^o, 315^o (Rej)$
$\theta = 116.6^o, 135^o\ (1\ d.p.)$

4 a) Given that $\cos 2x = a + b$ and $\sin 2x = a - b$. Show that $a^2 + b^2 = \frac{1}{2}$.
b) Find a function that has the following graph



a) $(a+b)^2 = \cos^2 2x$
$a^2 + b^2 + 2ab = \cos^2 2x \quad -(1)$
$(a-b)^2 = \sin^2 2x$
$a^2 + b^2 - 2ab = \sin^2 2x \quad -(2)$
$(1) + (2)$:
$2(a^2 + b^2) = \sin^2 2x + \cos^2 2x$
$2(a^2 + b^2) = 1$
$a^2 + b^2 = \frac{1}{2}\ (Shown)$

b) amplitude = 6
Period = $\frac{\pi}{2}$
$\frac{2\pi}{b} = \frac{\pi}{2}$
$b = 4$
Function: $y = |6\sin 4x| - 2$

5 Prove that $\sin^4 A + \cos^4 A = \frac{1}{4}(3 + \cos 4A)$.

$(Hint: Square(\sin^2 A + \cos^2 A)\ to\ help\ proof\ above\ equation)$

$(\sin^2 A + \cos^2 A)^2 = \sin^4 A + \cos^4 A + 2\sin^2 A \cos^2 A$

$\sin^4 A + \cos^4 A = (\sin^2 A + \cos^2 A)^2 - 2\sin^2 A \cos^2 A$

$LHS = \sin^4 A + \cos^4 A$

$= (\sin^2 A + \cos^2 A)^2 - 2\sin^2 A \cos^2 A$

$= 1 - \frac{4\sin^2 A \cos^2 A}{2}$

$= 1 - \frac{(2\sin A \cos A)^2}{2}$

$= 1 - \frac{(\sin 2A)^2}{2}$

$= \frac{4 - 2\sin^2 2A}{4}$

$= \frac{3 + \cos 4A}{4}$

$= \frac{1}{4}(3 + \cos 4A) = RHS\ (Shown)$

6 ABC is a triangle where $\tan\left(\frac{\angle A}{2}\right) = \frac{1}{2}$.

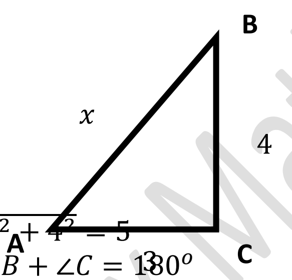
i) Show that $\tan \angle A = \frac{4}{3}$.

ii) Find the exact values of $\sin(\angle B + \angle C)$ and $\cos(\angle B + \angle C)$

(Hint: $\angle A + \angle B + \angle C = 180^o$)

i) $\tan \angle A = \frac{2\tan\left(\frac{\angle A}{2}\right)}{1 - \tan^2\left(\frac{\angle A}{2}\right)} = \frac{2\left(\frac{1}{2}\right)}{1 - \left(\frac{1}{2}\right)^2} = \frac{4}{3}$ (Shown)

ii)



$x = \sqrt{3^2 + 4^2} = 5$

$\angle A + \angle B + \angle C = 180^o$

$\angle B + \angle C = 180^o - \angle A$

$\sin(\angle B + \angle C) = \sin(180^o - \angle A)$

$= \sin(\angle A)$

$= \frac{4}{5}$

$\cos(\angle B + \angle C) = \cos(180^o - \angle A)$

$= -\cos(\angle A)$

$= -\frac{3}{5}$

7 The depth of water, y meters, at a particular coast, t hours after 12 am is given by:
$y = 4 + 3\sin\left(\frac{\pi}{6}t\right)$, where $0 \leq t \leq 24$
i) State the amplitude of y
ii) What are the depths of water at high tide and low tide?
iii) At what times of the day will low tide occur?

i) Amplitude $= 3$
ii) High tide $= 4 + 3 = 7\ m$
Low tide $= 4 - 3 = 1\ m$
iii) $4 + 3\sin\left(\frac{\pi}{6}t\right) = 1$
$\sin\left(\frac{\pi}{6}t\right) = -1$ (3rd and 4th Quadrants)
$\alpha = \frac{\pi}{2}$
$\frac{\pi}{6}t = \frac{3\pi}{2}, \frac{7\pi}{2}$
$t = 9, 21$
9 hours after 12am = 9am
21 hours after 12am = 9pm
Low tide occurs at $9am$ and $9pm$.

8 a) Prove the identity $\frac{\sin(A+B)}{\sin(A-B)} = \frac{\tan A + \tan B}{\tan A - \tan B}$

b) Prove the identity $\sin^3 x + \cos^3 x = (\sin x + \cos x)(1 - \sin x \cos x)$

a) LHS $= \frac{\sin(A+B)}{\sin(A-B)}$

$= \frac{\sin A \cos B + \sin B \cos A}{\sin A \cos B - \sin B \cos A}$

$= \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\sin B \cos A}{\cos A \cos B}}{\frac{\sin A \cos B}{\cos A \cos B} - \frac{\sin B \cos A}{\cos A \cos B}}$

$= \frac{\tan A + \tan B}{\tan A - \tan B}$

$= RHS$ (Shown)

b) $RHS = (\sin x + \cos x)(1 - \sin x \cos x)$

$= \sin x + \cos x - \sin^2 x \cos x - \sin x \cos^2 x$

$= \sin x + \cos x - (1 - \cos^2 x) \cos x - \sin x\,(1 - \sin^2 x)$

$= \sin x + \cos x - \cos x + \cos^3 x - \sin x + \sin^3 x$

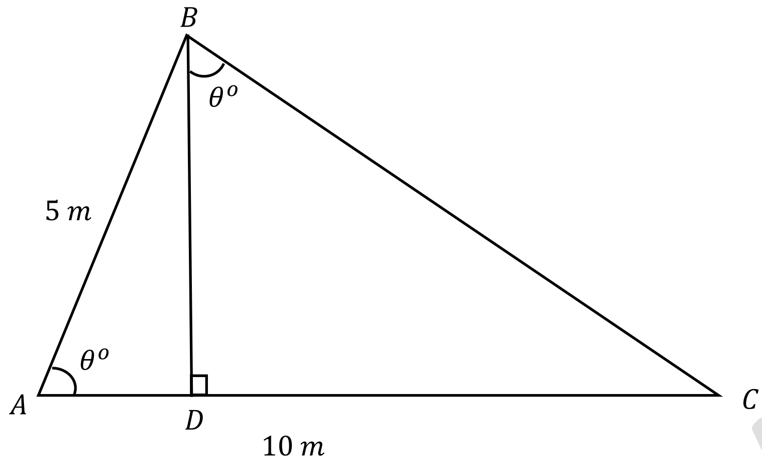
$= \cos^3 x + \sin^3 x$

$= LHS$ (Shown)

9 i) Prove the identity $\sin 2x - \tan x = \tan x \cos 2x$.
ii) Hence, without using a calculator, find the value of $\tan(67.5^o)$.

i) $RHS = \tan x \cos 2x$
$= \frac{\sin x}{\cos x}(2\cos^2 x - 1)$
$= 2\sin x \cos x - \frac{\sin x}{\cos x}$
$= \sin 2x - \tan x$
$= LHS$ (Proven)

ii) $\sin 2x - \tan x = \tan x \cos 2x$
$\sin 2x = \tan x (\cos 2x + 1)$
$\tan x = \frac{\sin 2x}{\cos 2x + 1}$
Sub $x = 67.5^o$
$\tan(67.5^o) = \frac{\sin 135^o}{\cos 135^o + 1}$
$= \frac{\sin(180^o - 45^o)}{\cos(180^o - 45^o) + 1}$
$= \frac{\sin 45^o}{-\cos 45^o + 1}$
$= \frac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}} + 1}$
$= \frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{2}-1}{\sqrt{2}}}$
$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}-1}$
$= \frac{1}{\sqrt{2}-1}$
$= \frac{1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}$
$= \sqrt{2} + 1$



10 In the diagram above, ABC is a triangle such that $\angle BAD = \angle DBC = \theta^o$, $AB = 5\,m$ and $AC = 10\,m$.

i) Find BC (Leave your answer in surd form)

ii) Show that $BD = 5\sin\theta^o$

iii) Show that $BD = 5\sqrt{5}\cos\theta^o$

iv) Hence, show that $2BD = 10\cos(\theta^o - 24.1^o)$

i) $\angle ABD = 90^o - \theta^o$

$\angle ABC = 90^o - \theta^o + \theta^o = 90^o$

By Pythagoras theorem, $BC^2 = AC^2 - AB^2$

$BC^2 = 10^2 - 5^2$

$BC = \sqrt{75} = 5\sqrt{5}$

ii) ΔABD is a right angled triangle,

$\sin\theta^o = \frac{BD}{5}$

$BD = 5\sin\theta^o$ (Shown)

iii) ΔBCD is a right angled triangle,

$\cos\theta^o = \frac{BD}{BC} = \frac{BD}{5\sqrt{5}}$

$BD = 5\sqrt{5}\cos\theta^o$ (Shown)

iv) $2BD = 5\sqrt{5}\cos\theta^o + 5\sin\theta^o = R\cos(\theta^o - \alpha)$

$R^2 = \left(5\sqrt{5}\right)^2 + 5^2$

$R = \sqrt{100} = 10$

$\tan\alpha = \frac{5}{5\sqrt{5}}$

$\alpha = 24.1^o$

$\therefore 2BD = 10\cos(\theta^o - 24.1^o)$ (Shown)