Surds and Indices

1 Find the value of k and m if $\frac{3^{3n+2}+3^{3n-1}}{2\times 3^{nk}} = m$. m is a constant and $n > 0$.

$\frac{3^{3n+2}+3^{3n-1}}{2\times 3^{nk}} = m$

$\frac{9(3^{3n})+\left(\frac{1}{3}\right)3^{3n}}{2\times 3^{nk}} = m$

$\frac{27(3^{3n})+3^{3n}}{3\times 2\times 3^{nk}} = m$

$\frac{28(3^{3n})}{6\times 3^{nk}} = m$

$\frac{28}{6} = m$ and $\frac{3^{3n}}{3^{nk}} = 3^0$

$m = 4\frac{2}{3}$ and $3n - nk = 0$

$n(3-k) = 0$

$n = 0$ (Rej) or $k = 3$

$\therefore k = 3$ and $m = 4\frac{2}{3}$

2 Without using a calculator, solve the equation $\left(\frac{3^{y+1}}{24}\right)^{\frac{1}{x}} = \sqrt{108}$.

$\left(\frac{3^{y+1}}{2^3\times 3}\right)^{\frac{1}{x}} = \sqrt{2^2\times 3^3}$

$3^{\frac{y+1}{x}-\frac{1}{x}}\times 2^{-\frac{3}{x}} = 2^1\times 3^{\frac{3}{2}}$

By comparing powers of 2:

$-\frac{3}{x} = 1$

$x = -3$

By comparing powers of 3:

$\frac{y+1}{x} - \frac{1}{x} = \frac{3}{2}$

$\frac{y}{x} = \frac{3}{2}$

$\frac{y}{-3} = \frac{3}{2}$

$y = -\frac{9}{2} = -4.5$

3 Find the values of a and b if $(2x^3)^a\left(\frac{1}{4x}\right)^{2-a}=\frac{ba^2}{x^{-2}}$

$(2x^3)^a\left(\frac{1}{4x}\right)^{2-a}=\frac{ba^2}{x^{-2}}$

$2^a\times x^{3a}\times 4^{a-2}\times x^{a-2}=ba^2x^2$

By comparing x-terms:

$x^{3a}\times x^{a-2}=x^2$

$3a+a-2=2$

$a=1$

By comparing coefficients:

$2^a\times 4^{a-2}=ba^2$

$2\times 4^{-1}=b$ (sub $a=1$)

$b=\frac{1}{2}$

4 Find the exact value of x is $32^x=\sqrt{4\sqrt{8\sqrt{32}}}$

$32^x=\sqrt{4\sqrt{8\sqrt{32}}}$

$2^{5x}=\sqrt{2^2\sqrt{2^3\sqrt{2^5}}}$

$2^{10x}=2^2\sqrt{2^3\sqrt{2^5}}$

$2^{10x-2}=\sqrt{2^3\sqrt{2^5}}$

$2^{20x-4}=2^3\sqrt{2^5}$

$2^{20x-7}=\sqrt{2^5}$

$2^{40x-14}=2^5$

$40x-14=5$

$x=\frac{19}{40}$

5 Simplify $\frac{5}{\sqrt{1}+\sqrt{2}}+\frac{5}{\sqrt{2}+\sqrt{3}}+\cdots+\frac{5}{\sqrt{14}+\sqrt{15}}+\frac{5}{\sqrt{15}+\sqrt{16}}$

$\frac{5}{\sqrt{1}+\sqrt{2}}+\frac{5}{\sqrt{2}+\sqrt{3}}+\cdots+\frac{5}{\sqrt{14}+\sqrt{15}}+\frac{5}{\sqrt{15}+\sqrt{16}}$

Rationalize denominator:

$\frac{5\sqrt{1}-5\sqrt{2}}{1-2}+\frac{5\sqrt{2}-5\sqrt{3}}{2-3}+\cdots+\frac{5\sqrt{14}-5\sqrt{15}}{14-15}+\frac{5\sqrt{15}-5\sqrt{16}}{15-16}$

$=-5+5\sqrt{2}-5\sqrt{2}+5\sqrt{3}+\cdots-5\sqrt{14}+5\sqrt{15}-5\sqrt{15}+5\sqrt{16}$

$=-5+5\sqrt{16}$

$=-5+20$

$=15$

6 Solve $\sqrt{y-4}+\sqrt{3y+1}=5$

$\sqrt{y-4}+\sqrt{3y+1}=5$

$\left(\sqrt{y-4}+\sqrt{3y+1}\right)^2=5^2$

$(y-4)+2\sqrt{y-4}\sqrt{3y+1}+(3y+1)=25$

$2\sqrt{(y-4)(3y+1)}=-4y+28$

$\sqrt{(y-4)(3y+1)}=-2y+14$

$(y-4)(3y+1)=(-2y+14)^2$

$3y^2-11y-4=4y^2-56y+196$

$y^2-45y+200=0$

$(y-5)(y-40)=0$

$y=5$ or $y=40$ (Rejected since does not hold true when substituted back to original equation)

7 Given that $5(3^{3x-2})+4(2^{2x+2})=4^x$, express $\left(\frac{4}{27}\right)^x$ in the form $\frac{a}{b}$ where a and b are integers.

$5(3^{3x-2})+4(2^{2x+2})=4^x$
$\frac{5}{9}(27^x)+16(4^x)=4^x$
$\frac{5}{9}(27^x)=-15(4^x)$
$\frac{5}{9(-15)}=\frac{4^x}{27^x}$
$-\frac{1}{27}=\left(\frac{4}{27}\right)^x$

8 Without using a calculator, find the values of a, b and c for which the solution of the equation $x\sqrt{20}=\sqrt{24}+x\sqrt{6}$ is $\frac{a+b\sqrt{c}}{7}$.

$x\sqrt{20}=\sqrt{24}+x\sqrt{6}$
$x(\sqrt{20}-\sqrt{6})=\sqrt{24}$
$x=\frac{\sqrt{24}}{\sqrt{20}-\sqrt{6}}$
$x=\frac{\sqrt{24}}{\sqrt{20}-\sqrt{6}}\times\frac{\sqrt{20}+\sqrt{6}}{\sqrt{20}+\sqrt{6}}$
$=\frac{\sqrt{2^3\times3}(\sqrt{2^2\times5}+\sqrt{2\times3})}{14}$
$=\frac{\sqrt{2^5\times3\times5}+\sqrt{2^4\times3^2}}{14}$
$=\frac{4\sqrt{2\times3\times5}+12}{14}$
$=\frac{6+2\sqrt{30}}{7}$
$a=6$, $b=2$ and $c=30$.

9 Show that $3^{n+3}+3^n-3^{n+2}$ is exactly divisible by 19 for all positive integer values of n.

$3^{n+3}+3^n-3^{n+2}=3^n(27)+3^n(1)-3^n(9)$
$=3^n(27+1-9)$
$=3^n(19)$
$\therefore$ It is divisible by 19 for all positive integer values of n.

10 Solve $2(3^x)+5\sqrt{3^x}=3$
$2(3^x)+5\sqrt{3^x}-3=0$
$2\left(3^{\frac{x}{2}}\right)^2+5\left(3^{\frac{x}{2}}\right)-3=0$
Let $y=3^{\frac{x}{2}}$,
$2y^2+5y-3=0$
$(2y-1)(y+3)=0$
$y=\frac{1}{2}$ or $y=-3$
$3^{\frac{x}{2}}=\frac{1}{2}$ or $3^{\frac{x}{2}}=-3$
$\frac{x}{2}\lg 3=\lg\frac{1}{2}$ or $\frac{x}{2}\lg 3=\lg -3$ (N.A.)
$\frac{x}{2}=\frac{\lg\frac{1}{2}}{\lg 3}$
$x=\frac{2\lg\frac{1}{2}}{\lg 3}$
$x=-1.26\ (3\ s.f.)$

11 a) Find the value of x given that $\sqrt{(x+8)\sqrt{(x+8)\sqrt{(x+8)}}} = 2^{\frac{7}{2}}$

b) Given that $x = 3 + 2\sqrt{2}$, find the value of $\sqrt{x} + \frac{1}{\sqrt{x}}$.

a) $\sqrt{(x+8)\sqrt{(x+8)\sqrt{(x+8)}}} = 2^{\frac{7}{2}}$

$(x+8)\sqrt{(x+8)\sqrt{(x+8)}} = 2^7$

$\sqrt{(x+8)\sqrt{(x+8)}} = \frac{2^7}{x+8}$

$(x+8)\sqrt{(x+8)} = \frac{2^{14}}{(x+8)^2}$

$\sqrt{x+8} = \frac{2^{14}}{(x+8)^3}$

$x+8 = \frac{2^{28}}{(x+8)^6}$

$(x+8)^7 = 2^{28}$

$x+8 = 2^4$

$x = 16 - 8 = 8$

b) $\sqrt{x} + \frac{1}{\sqrt{x}} = \frac{x+1}{\sqrt{x}}$

$= \frac{3+2\sqrt{2}+1}{\sqrt{3+2\sqrt{2}}}$

$= \frac{4+2\sqrt{2}}{\sqrt{3+2\sqrt{2}}}$

$= \left(\frac{(4+2\sqrt{2})^2}{3+2\sqrt{2}}\right)^{\frac{1}{2}}$

$= \left(\frac{16+8+16\sqrt{2}}{3+2\sqrt{2}}\right)^{\frac{1}{2}}$

$= \left(\frac{24+16\sqrt{2}}{3+2\sqrt{2}} \times \frac{3-2\sqrt{2}}{3-2\sqrt{2}}\right)^{\frac{1}{2}}$

$= \left(\frac{72+48\sqrt{2}-48\sqrt{2}-64}{9-8}\right)^{\frac{1}{2}}$

$= (8)^{\frac{1}{2}}$

$= 2\sqrt{2}$