Polynomials

1 Without using long division, find the remainder when $2\,x^6 + x^4 - 15\,x^2 - 14$ is divided by $x^2 + 2$.

Sub $x^2 = y$:
$f(y) = 2y^3 + y^2 - 15y - 14$
Divided by $y + 2$.
By Remainder Theorem,
$f(-2) = 2(-2)^3 + (-2)^2 - 15(-2) - 14$
$= 4$
∴ The remainder 4.

2 A cubic polynomial, $f(x)$, leaves a remainder of 12 when divided by x and $f(x+1) - f(x-1) \equiv 12x^2 - 12x - 42$.By substituting suitable values of x,
a) Find the remainder when $f(x)$ is divided by $(x - 2)$
b) Show that $f(-2) = 30$
c) Show that $(x - 4)$ is a factor of $f(x)$.

a) Since $f(x)$ leaves a remainder of 12 when divided by x,
$f(0) = 12$

Sub $x = 1$:
$f(1+1) - f(1-1) = 12(1)^2 - 12(1) - 42$
$f(2) - f(0) = 12 - 12 - 42$
$f(2) - 12 = 12 - 12 - 42$
$f(2) = -30$
The remainder when $f(x)$ is divided by $(x - 2)$ is -30.

b) Sub $x = -1$:
$f(-1+1) - f(-1-1) = 12(-1)^2 - 12(-1) - 42$
$f(0) - f(-2) = 12 + 12 - 42$
$12 - f(-2) = 12 + 12 - 42$
$f(-2) = 30$ (Shown)

c) Sub $x = 3$:
$f(3+1) - f(3-1) = 12(3)^2 - 12(3) - 42$
$f(4) - f(2) = 108 - 36 - 42$
$f(4) - (-30) = 108 - 36 - 42$
$f(4) = 0$
By Factor theorem, since $f(4) = 0$, $(x - 4)$ is a factor of $f(x)$.

3 Given that $(x-1)(x-2)(Ax+B)+C(x-2)+D=3\,x^3-7\,x^2+3\,x+2$ for all values of x, find A, B, C and D.

By comparing x^3 coefficients:
$A=3$
Sub $x=2$:
$D=3(2)^3-7(2)^2+3(2)+2=4$
Sub $x=1$:
$-C+D=3-7+3+2$
$C=3$
Comparing x-independent term:
$2B-2C+D=2$
$B=2$

4 $(x-2)$ is a factor of $g(x)+5$, where $g(x)$ is a polynomial. Find the remainder when $f(x)=(2x^3+3x^2-4)g(x)$ is divided by $(x-2)$.

By Factor Theorem: $g(2)+5=0$
$g(2)=-5$
By Remainder Theorem:
Remainder $=f(2)=(2(2)^3+3(2)^2-4)g(2)$
$=(24)(-5)$
$=-120$

5 The term containing the highest power of x in the polynomial $f(x)$ is x^4 and the roots of $f(x) = 0$ are -6 and 3. $f(x)$ has a remainder of -84 when divided by $(x-1)$ and a remainder of -96 when divided by $(x-2)$. Find the expression for $f(x)$.

$f(x) = (x-3)(x+6)(x+a)(x+b)$
$f(1) = (1-3)(1+6)(1+a)(1+b)$
$-84 = (-14)(1+a)(1+b)$
$b = -\frac{84}{(-14)(1+a)} - 1 \qquad -(1)$
$f(2) = (2-3)(2+6)(2+a)(2+b)$
$-96 = -8(2+a)(2+b) \qquad -(2)$
Sub (1) into (2):
$-96 = -8(2+a)\left(2 - \frac{84}{(-14)(1+a)} - 1\right)$
$12 = (2+a)\left(1 + \frac{6}{(1+a)}\right)$
$12 = (2+a)(\frac{1+a+6}{1+a})$
$12(1+a) = (2+a)(a+7)$
$12 + 12a = 2a + a^2 + 7a + 14$
$a^2 - 3a + 2 = 0$
$(a-1)(a-2) = 0$
$a = 1 \text{ or } a = 2$

6 Express $\frac{3x^2+5}{x^4-1}$ in partial fractions

$x^4 - 1^4 = (x^2 - 1)\,(x^2 + 1)$

$= (x + 1)(x - 1)(x^2 + 1)$

$\frac{3x^2+5}{x^4-1} = \frac{3x^2+5}{(x+1)(x-1)(x^2+1)}$

Let $\frac{3x^2+5}{(x+1)(x-1)(x^2+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}$

$3x^2 + 5 = A(x^2 + 1)(x - 1) + B(x + 1)(x^2 + 1) + (Cx + D)(x + 1)(x - 1)$

Let $x = -1, -4A = 8$

$A = -2$

Let $x = 1, 4B = 8$

$B = 2$

Let $x = 0, -A + B - D = 5$

$D = -1$

Comparing x^3 coefficient, $A + B + C = 0$

$C = 0$

$\therefore \frac{3x^2+5}{x^4-1} = \frac{-2}{x+1} + \frac{2}{x-1} + \frac{-1}{x^2+1}$

7 Given that $f(x) = 4x^3 - 2x^2 + 5x - 1$, find
i) the remainder when $f(x)$ is divided by $(x - 1)$
ii) the remainder when $f(x - 8)$ is divided by $(x - 9)$.
iii) deduce the remainder when $f(x^2 - 6)$ is divided by $(x^2 - 8)$.

i) $f(1) = 4(1)^3 - 2(1)^2 + 5(1) - 1$
$= 6$

ii) $f(x - 8) = 4(x - 8)^3 - 2(x - 8)^2 + 5(x - 8) - 1$
When $f(x - 8)$ is divided by $(x - 9)$,
$f(9 - 8) = 4(9 - 8)^3 - 2(9 - 8)^2 + 5(9 - 8) - 1$
$f(1) = 4(1)^3 - 2(1)^2 + 5(1) - 1$
$= 6$

iii) $f(x^2 - 6) = 4(x^2 - 6)^3 - 2(x^2 - 6)^2 + 5(x^2 - 6) - 1$
$f(8 - 6) = 4(8 - 6)^3 - 2(8 - 6)^2 + 5(8 - 6) - 1$
$f(2) = 4(2)^3 - 2(2)^2 + 5(2) - 1$
$= 33$

8 When the function $f(x)$ is divided by $(x + 1)$, the remainder is -5. When $f(x)$ is divided by $(x - 1)$, the remainder is -1. When $f(x)$ is divided by $(x^2 - 1)$, the remainder is $(Ax + B)$. Find A and B.

$f(x) = (x^2 - 1)Q(x) + Ax + B$
$f(x) = (x - 1)(x + 1)Q(x) + Ax + B$
When $f(x)$ is divided by $(x + 1)$, remainder $= -5$.
$f(-1) = -5$
$-A + B = -5$ -(1)
When $f(x)$ is divided by $(x - 1)$, remainder $= -1$.
$f(1) = -1$
$A + B = -1$ -(2)
Simultaneous solve (1) and (2), $A = 2$, $B = -3$

9 $f(x)$ is a function where $f(x) = ax^3 + bx^2 + 2x - 5$. $2f(x) - 6$ is divisible by $(x - 1)$ and when $f(x) + 4$ is divided by $(x + 2)$, it leaves a remainder of -5. Find A and B.

$2f(x) - 6 = 2ax^3 + 2bx^2 + 4x - 10 - 6$

Given that $2f(x) - 6$ is divisible by $(x - 1)$,
$2f(1) - 6 = 0$
$2a(1)^3 + 2b(1)^2 + 4(1) - 16 = 0$
$2a + 2b - 12 = 0$ -(1)

Given that $f(x) + 4$ leaves a remainder of -5 when divided by $(x + 2)$,
$f(-2) + 4 = -5$
$a(-2)^3 + b(-2)^2 + 2(-2) - 5 + 4 = -5$
$-8a + 4b - 4 - 5 + 4 = -5$
$-8a + 4b = 0$ -(2)

Simultaneously solve (1) and (2), $a = 2$ and $b = 4$.

10 Given that (x^2-3) is a factor of $f(x)=x^3+ax^2+bx-3$
i) Find the value of a and b.
ii) Hence, factorize $f(x)$ completely.
iii) Hence, solve the equation $1+ay+by^2-3y^3=0$.

i) Let $x^2-3=0$,
$x=\pm\sqrt{3}$

By factor theorem,
$f(\sqrt{3})=0$ and $f(-\sqrt{3})=0$.

$f(\sqrt{3})=0$
$(\sqrt{3})^3+a(\sqrt{3})^2+b\sqrt{3}-3=0$
$3\sqrt{3}+3a+b\sqrt{3}-3=0$ -(1)

$f(-\sqrt{3})=0$
$(-\sqrt{3})^3+a(-\sqrt{3})^2+b(-\sqrt{3})-3=0$
$-3\sqrt{3}+3a-b\sqrt{3}-3=0$ -(2)

$(1)+(2)$: $6a-6=0$
$a=1$
Sub $a=1$ into (1): $3\sqrt{3}+3(1)+b\sqrt{3}-3=0$
$b=-3$

ii) $x^3+x^2-3x-3=(x^2-3)(Ax+B)$
Comparing coefficient: $A=1, B=1$
$\therefore x^3+x^2-3x-3=(x+\sqrt{3})(x-\sqrt{3})(x+1)$

iii) $1+y-3y^2-3y^3=0$
$\frac{1}{y^3}+\frac{y}{y^3}-\frac{3y^2}{y^3}-\frac{3y^3}{y^3}=\frac{0}{y^3}$ (divide both sides by y^3)
$(y^{-1})^3+(y^{-1})^2-3(y^{-1})-3=0$
From part ii), $x^3+x^2-3x-3=(x+\sqrt{3})(x-\sqrt{3})(x+1)$
Sub $x=y^{-1}$:
$(y^{-1})^3+(y^{-1})^2-3(y^{-1})-3=(y^{-1}+\sqrt{3})(y^{-1}-\sqrt{3})(y^{-1}+1)$
Hence, $y^{-1}=-\sqrt{3}$ or $y^{-1}=\sqrt{3}$ or $y^{-1}=-1$
$\therefore y=-\frac{1}{\sqrt{3}}$ or $y=\frac{1}{\sqrt{3}}$ or $y=-1$