Logarithms

1 Solve $log_{81}[log_3(26+x)] = \frac{log_{25} 5}{2}$

$log_{81}[log_3(26+x)] = \frac{log_{25} 5}{2}$

$log_{81}[log_3(26+x)] = \frac{log_{25} 25^{\frac{1}{2}}}{2}$

$log_{81}[log_3(26+x)] = \frac{1}{4}$

$log_3(26+x) = 81^{\frac{1}{4}}$

$log_3(26+x) = 3$

$(26+x) = 3^3$

$26 + x = 27$

$x = 1$

2 Solve $log_{\frac{x}{4}} x + log_{4x} x = -1$

$\frac{1}{log_x \frac{x}{4}} + \frac{1}{log_x 4x} = -1$

$\frac{1}{log_x x - log_x 4} + \frac{1}{log_x x + log_x 4} = -1$

$\frac{1}{1 - log_x 4} + \frac{1}{1 + log_x 4} = -1$

$Let\ y = log_x 4$

$\frac{1}{1-y} + \frac{1}{1+y} = -1$

$\frac{1+y+1-y}{(1-y)(1+y)} = -1$

$\frac{2}{(1-y^2)} = -1$

$2 = -1 + y^2$

$y^2 = 3$

$y = \pm\sqrt{3}$

$log_x 4 = +\sqrt{3}$ or $log_x 4 = -\sqrt{3}$

$4 = x^{\sqrt{3}}$ or $4 = x^{-\sqrt{3}}$

$x = 4^{\frac{1}{\sqrt{3}}}$ or $x = 4^{-\frac{1}{\sqrt{3}}}$

$x = 2.23$ or $x = 0.449$ (3s.f.)

3 Solve $log_3 e^x - log_3\left(\frac{1}{2}\right) - log_9(4e^x + 3) = 0$

$log_3 e^x - log_3\left(\frac{1}{2}\right) - log_9(4e^x + 3) = 0$

$log_3 2e^x - \frac{log_3(4e^x+3)}{log_3 9} = 0$

$log_3 2e^x - \frac{log_3(4e^x+3)}{2} = 0$

$log_3 2e^x - log_3(4e^x + 3)^{\frac{1}{2}} = 0$

$log_3 \frac{2e^x}{(4e^x+3)^{\frac{1}{2}}} \quad = 0$

$\frac{2e^x}{(4e^x+3)^{\frac{1}{2}}} = 1$

$2e^x = (4e^x + 3)^{\frac{1}{2}}$

$4e^{2x} = 4e^x + 3$

Let $y = e^x$

$4y^2 - 4y - 3 = 0$

$(2y - 3)(2y + 1) = 0$

$y = \frac{3}{2} \quad or \quad -\frac{1}{2}$

$e^x = \frac{3}{2} \quad or \quad -\frac{1}{2} \ (rej)$

$x = \ln(\frac{3}{2})$

4 Show that $(log_2 3)\,(log_9 16) = 2$

$(log_2 3)\,(log_9 16) = \ (log_2 3) \times 4(log_9 2)$

$= 4(log_2 3)\,\frac{log_2 2}{log_2 9}$

$= 4(log_2 3)\,\frac{1}{2log_2 3}$

$= 4(\cancel{log_2 3})\,\frac{1}{2\,\cancel{log_2 3}}$

$= 2$ (Shown)

5 Prove that $e^{\ln x} = x$

Let $e^{\ln x} = y$
$\ln e^{\ln x} = \ln y$
$\ln x\,(\ln e) = \ln y$
$\ln x = \ln y$
$x = y$
$Since\ x = y\ and\ y = e^{\ln x}, therefore\ x = e^{\ln x}$

6 Solve $log_2(log_2(log_2 x)) = 1$
$log_2(log_2(log_2 x)) = 1$
$log_2(log_2 x) = 2$
$log_2 x = 2^2 = 4$
$x = 2^4$
$x = 16$

7 Given that $x^2 + y^2 = 14xy$, show that $2\lg\left(\frac{x+y}{4}\right) = \lg x + \lg y$

$x^2 + y^2 = 14xy$
$x^2 + y^2 + 2xy = 14xy + 2xy$
$(x + y)^2 = 16xy$
$\frac{(x+y)^2}{4^2} = xy$
$\left(\frac{x+y}{4}\right)^2 = xy$
$2\lg\left(\frac{x+y}{4}\right) = \lg(xy)$
$2\lg\left(\frac{x+y}{4}\right) = \lg x + \lg y$

8 Given that $\log_k a = x$ and $\log_k b = y$, express the following in terms of x and y.
i) $\log_{ab} k^2$
ii) $\log_b \left(\frac{ab^2}{k}\right)$

i) $\log_{ab} k^2 = 2\log_{ab} k$
$= 2\left[\frac{\log_k k}{\log_k ab}\right]$
$= 2\left(\frac{1}{\log_k a + \log_k b}\right)$
$= \frac{2}{x+y}$

ii) $\log_b \left(\frac{ab^2}{k}\right) = \frac{\log_k\left(\frac{ab^2}{k}\right)}{\log_k b}$
$= \frac{\log_k a + 2\log_k b - \log_k k}{\log_k b}$
$= \frac{x+2y-1}{y}$

9 Without using a calculator, evaluate $(\lg 5)^2 + \lg 2 \lg 50$

$(\lg 5)^2 + \lg 2 \lg 50 = (\lg 5)^2 + \lg 2\,(\lg 5 + \lg 10)$
$= (\lg 5)^2 + \lg 2 \lg 5 + \lg 2\,(\lg 10)$
$= \lg 5\,(\lg 5 + \lg 2) + \lg 2$
$= \lg 5\,(\lg 10) + \lg 2$
$= \lg 5 + \lg 2$
$= \lg 10$
$= 1$

10 If $a = \log_x 99$ and $b = \log_x 363$, express the following in terms of a and b.
i) $\log_x 9$
ii) $\log_x 33$

$a = \log_x 99$
$a = \log_x 3^2 \times 11$
$a = 2\log_x 3 + \log_x 11$ -(1)
$b = \log_x 363$
$b = \log_x 3 \times 11^2$
$b = \log_x 3 + 2\log_x 11$ -(2)

2×(2)-(1):
$2b - a = 3\log_x 11$
$\log_x 11 = \frac{2b-a}{3}$

2×(1)-(2):
$2a - b = 3\log_x 3$
$\log_x 3 = \frac{2a-b}{3}$

$\log_x 9 = 2\log_x 3$
$= 2\left(\frac{2a-b}{3}\right)$
$= \frac{4a-2b}{3}$

ii) $\log_x 33 = \log_x 3 + \log_x 11$
$= \frac{2a-b}{3} + \frac{2b-a}{3}$
$= \frac{a+b}{3}$

11 Solve the equation $\ln(2xe - e) = 1 + \frac{1}{log_x e}$.

$\ln(2xe - e) = \ln(e) + \log_e x$
$\ln(2xe - e) = \ln(ex)$
$2xe - e = xe$
$x(2e - e) = e$
$x = 1$