## (1) Simultaneous, Quadrilaterals and Inequalities

1 Find the value(s) of $k$ for the following simultaneous equations, given that the equations have no solution.

$(k+1)y = (2k-1)x + 5$ ---(1)

$4y = (k+2)x + 10$ ---(2)

If both equations have no solution, the 2 lines are parallel and their $y$-intercepts are different.
Therefore, $m_1 = m_2$ and $c_1 \neq c_2$

From (1) $y = \frac{2k-1}{k+1}x + \frac{5}{k+1}$

From (2) $y = \frac{k+2}{4}x + \frac{10}{4}$

$\frac{2k-1}{k+1} = \frac{k+2}{4}$

$8k - 4 = (k+2)(k+1)$

$8k - 4 = k^2 + 3k + 2$

$k^2 - 5k + 6 = 0$

$(k-2)(k-3) = 0$

$k = 2$ or $k = 3$

When $k = 2$:

Eqn (1): $3y = 3x + 5$

$y = x + \frac{5}{3}$

Eqn(2): $4y = 4x + 10$

$y = x + \frac{10}{4}$

Since $c_1 \neq c_2$, $k = 2$ is a valid solution.

When $k = 3$:

Eqn (1): $4y = 5x + 5$

$y = \frac{5}{4}x + \frac{5}{4}$

Eqn(2): $4y = 5x + 10$

$y = \frac{5}{4}x + \frac{10}{4}$

Since $c_1 \neq c_2$, $k = 3$ is a valid solution.

Therefore the values of $k$ are 2 and 3.

2 The equation $2x^2 + 8x = 1$ has roots $\alpha$ and $\beta$.

a) State the value of $\alpha + \beta$ and $\alpha\beta$

b) Find the value of $\alpha^2 - \beta^2$, leaving your answer in surd form.

c) Find the quadratic equation whose roots are $\alpha^4 - \beta^4$

a) $2x^2 + 8x - 1 = 0$

$\alpha + \beta = -4$

$\alpha\beta = -\frac{1}{2}$

b) $(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$ -(1)

$(\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta$

$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 2\alpha\beta - 2\alpha\beta$ [Substitute (1)]

$(\alpha - \beta) = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$ -(2)

$\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta)$

$\alpha^2 - \beta^2 = (\alpha + \beta)\sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$ [Substitute (2)]

$\alpha^2 - \beta^2 = (-4)\sqrt{(-4)^2 - 4\left(-\frac{1}{2}\right)}$

$\alpha^2 - \beta^2 = -4\sqrt{18}$

$= -12\sqrt{2}$

c) $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$

$= [(\alpha + \beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2$

$= \left[(-4)^2 - 2\left(-\frac{1}{2}\right)\right]^2 - 2\left(-\frac{1}{2}\right)^2$

$= 288.5$

$\alpha^4\beta^4 = \left(-\frac{1}{2}\right)^4$

$= \frac{1}{16}$

Quadratic equation is : $x^2 - 288.5x + \frac{1}{16} = 0$

$16x^2 - 4616x + 1 = 0$

3 It is given that $\alpha$ and $\beta$ are the roots of the equation $y = x^2 - x - 1$, where $\beta > \alpha$ and that $\alpha + \frac{1}{\alpha}$ and $\beta + \frac{1}{\beta}$ are the roots of another quadratic equation with integer coefficients. Without solving the values of $\alpha$ and $\beta$, find the exact value of $\alpha + \frac{1}{\alpha}$.

$\alpha + \beta = -\left(-\frac{1}{1}\right) = 1$
$\alpha\beta = -1$

$\alpha + \frac{1}{\alpha} + \beta + \frac{1}{\beta} = \alpha + \beta + \frac{1}{\alpha} + \frac{1}{\beta}$
$= \alpha + \beta + \frac{\beta+\alpha}{\alpha\beta}$
$= (1) + \frac{1}{-1}$
$= 0$

$\left(\alpha + \frac{1}{\alpha}\right)\left(\beta + \frac{1}{\beta}\right) = \alpha\beta + \frac{\beta}{\alpha} + \frac{\alpha}{\beta} + \frac{1}{\alpha\beta}$
$= \alpha\beta + \frac{\beta^2+\alpha^2}{\alpha\beta} + \frac{1}{\alpha\beta}$
$= \alpha\beta + \frac{(\beta+\alpha)^2-2\alpha\beta}{\alpha\beta} + \frac{1}{\alpha\beta}$
$= (-1) + \frac{(1)^2-2(-1)}{-1} + \frac{1}{-1}$
$= -1 - 3 - 1$
$= -5$

$\therefore$ Equation with $\alpha + \frac{1}{\alpha}$ and $\beta + \frac{1}{\beta}$ as roots is: $y = x^2 - 0x - 5$
$y = x^2 - 5$
When $y = 0$, $x^2 = 5$
$x = \pm\sqrt{5}$

Since $\beta > \alpha$ , $\alpha + \frac{1}{\alpha}$ is the smaller root.
$\therefore \alpha + \frac{1}{\alpha} = -\sqrt{5}$

4 a) If one root of the equation $4x^2 - 22x + k = 0$ is ten times the other, find the value of $k$.
b) Show that $2 - x^2 + 3x$ can never be greater than 5.

a) Let the roots be $\alpha$ and $10\alpha$,
Sum of roots $= -\frac{b}{a}$
$\alpha + 10\alpha = \frac{22}{4}$
$11\alpha = \frac{22}{4}$
$\alpha = \frac{2}{4} = 0.5$

Product of roots $= \frac{c}{a}$
$\alpha(10\alpha) = \frac{k}{4}$
$10\alpha^2 = \frac{k}{4}$
$10(0.5)^2 = \frac{k}{4}$
$k = 10$

b) $y = 2 - x^2 + 3x$ -(1)
$y = 5$ -(2)

Equate (1) with (2):
$2 - x^2 + 3x = 5$
$-x^2 + 3x - 3$

$b^2 - 4ac = 3^2 - 4(-1)(-3)$
$= -3$

Since $a < 0$, the graph is n-shaped
Since $b^2 - 4ac < 0$, the curve never touches the line $y = 5$
Hence, $2 - x^2 + 3x$ will never be greater than 5.

5 Show that the roots of the equation $x^2 + (2-k)x = \frac{3}{2}k$ are real for all real values of $k$.

$x^2 + (2-k)x - \frac{3}{2}k = 0$ - (1)

To prove that the equation $x^2 + (2-k)x - \frac{3}{2}k = 0$ has real roots for all real values of $k$,
we need to prove that the coefficient of $x^2$ term is positive (already proved by observation) and for Equation (1), the $b^2 - 4ac > 0$ .

$b^2 - 4ac = (2-k)^2 - 4(1)\left(-\frac{3}{2}k\right)$
$= 4 + k^2 - 4k + 6k$
$= k^2 + 2k + 4$ -(2)

To prove that $k^2 + 2k + 4 > 0$ for all values of $k$, we need to prove that for Equation (2), $b^2 - 4ac < 0$
Equation (2): $b^2 - 4ac = 2^2 - 4(1)(4) = -12 < 0$
Since the coefficient of $k^2$ is positive and that $b^2 - 4ac < 0$ for Equation (2),
$k^2 + 2k + 4 > 0$ for all values of $k$
Since for Equation (1), $b^2 - 4ac > 0$, it has real roots for all real values of $k$.

6 The roots of the equation $x^2 - 4x + k$ differs by $2s$. Show that $s^2 = 4 - k$. Given also that the roots are positive integers and that $k$ is a positive integer, find the possible values of $s$.

Let the roots be $\alpha$, $\alpha - 2s$.
Sum of roots $= -\frac{b}{a}$
$\alpha + \alpha - 2s = 4$
$\alpha = 2 + s$ -(1)
Product of roots $= \frac{c}{a}$
$\alpha(\alpha - 2s) = k$ -(2)
Sub (1) into (2):
$(2 + s)(2 + s - 2s) = k$
$(2 + s)(2 - s) = k$
$4 - s^2 = k$
$s^2 = 4 - k$ (Shown)

$x^2 - 4x + k = 0$
$x = \frac{4 \pm \sqrt{4^2 - 4k}}{2}$
$x = 2 \pm \sqrt{4 - k}$
Given that roots are positive integers and $k$ is a positive integer,
If $k = 0$, $x = 4$ or $0$ (Rej since 0 is not positive integer)
If $k = 1$, $x = 2 \pm \sqrt{3}$ (Rej since roots are not positive integer)
If $k = 2$, $x = 2 \pm \sqrt{2}$ (Rej since roots are not positive integer)
If $k = 3$, $x = 1$ or $3$
If $k = 4$, $x = 2$
If $k > 5$, there will be no real roots
When $k = 3$, $s = \pm 1$
When $k = 4$, $s = 0$
Possible values of $s$ are $-1, 0$ and $1$.

7 Given that $\alpha$ and $\beta$ are the roots of the equation $x^2 = x - 5$, prove that

a) $\frac{1-\alpha}{5} = \frac{1}{\alpha}$

b) $\alpha^3 + 4\alpha + 5 = 0$

a)

Since $\alpha$ is a root, $\alpha^2 = \alpha - 5$ -(1)

$\alpha^2 - \alpha = -5$

$\alpha(\alpha - 1) = -5$

$\frac{\alpha-1}{-5} = \frac{1}{\alpha}$

$\frac{1-\alpha}{5} = \frac{1}{\alpha}$ (Proved)

b) LHS $= \alpha^3 + 4\alpha + 5$

$= \alpha^2\alpha + 4\alpha + 5$

$= (\alpha - 5)\alpha + 4\alpha + 5$ (Substitute (1))

$= \alpha^2 - 5\alpha + 4\alpha + 5$

$= \alpha - 5 - \alpha + 5$ (Substitute (1))

$= 0$

$= RHS$ (Proved)

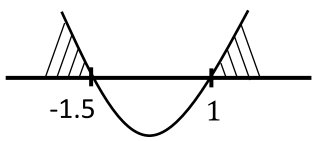
8 a) Find the range of values of $x$ for which $2x^2 + x - 6$ lies between $-3$ and $4$.
b) Show that if the roots of the equation $2x^2 + 3x - 2 + m(x-1)^2 = 0$ are real, then $m$ cannot be greater than $\frac{25}{12}$.

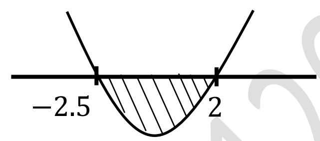
a) $-3 < 2x^2 + x - 6 < 4$

$-3 < 2x^2 + x - 6$ and $2x^2 + x - 6 < 4$

$2x^2 + x - 3 > 0$ and $2x^2 + x - 10 < 0$

$(x-1)(2x+3) > 0$ and $(x-2)(2x+5) < 0$




$x < -1.5$ or $x > 1$ and $x > -2.5$ or $x < 2$

$\therefore\ -2.5 < x < -1.5$ and $1 < x < 2$

b) $2x^2 + 3x - 2 + m(x-1)^2 = 0$

$2x^2 + 3x - 2 + mx^2 + m - 2xm = 0$

$(2+m)x^2 + (3-2m)x + m - 2 = 0$

If roots are real, $b^2 - 4ac \geq 0$

$(3-2m)^2 - 4(2+m)(m-2) \geq 0$

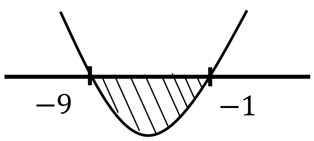
$9 + 4m^2 - 12m - 4m^2 + 16 \geq 0$

$-12m + 25 \geq 0$

$m \leq \frac{25}{12}$

9 Find the range of values of $k$ for which the graph of $y = kx^2 - 3x + kx$ lies entirely above the line $y = 4$.

$y = kx^2 + (k-3)x$ -(1)
$y = 4$ -(2)
Sub (1) into (2):
$4 = kx^2 + (k-3)x$
$kx^2 + (k-3)x - 4 = 0$
If the graph lies entirely above line,
$b^2 - 4ac < 0$ and $a > 0$
$(k-3)^2 - 4(k)(-4) < 0$ and $k > 0$ $(since\ a = k)$
$k^2 + 9 - 6k + 16k < 0$ and $k > 0$
$(k+1)(k+9) < 0$ and $k > 0$




$\boldsymbol{-9 < k < -1}$ **and** $\boldsymbol{k > 0}$
Therefore, there are <u>no real values of $k$</u> for which the graph lies entirely above the line.

10 i) Show that the expression $x^2 - x + \frac{7}{2}$ is always positive for all real values of $x$.

ii) Hence, find the values of $k$ which satisfy the inequality $\frac{-x^2+kx+2}{-(x^2-x+3.5)} < 2$ for all real values of $x$.

i) $x^2 - x + \frac{7}{2}$

$b^2 - 4ac = (-1)^2 - 4(1)(\frac{7}{2})$

$= -13$

Since $a > 0$ and $b^2 - 4ac < 0$, the curve is always positive.

ii) $\frac{-x^2+kx+2}{-(x^2-x+3.5)} < 2$

$-x^2 + kx + 2 > -2x^2 + 2x - 7$

$x^2 + (k-2)x + 9 > 0$

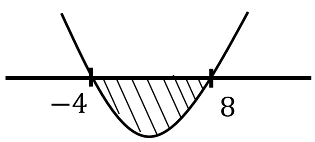
$b^2 - 4ac < 0$

$(k-2)^2 - 4(1)(9) < 0$

$k^2 + 4 - 4k - 36 < 0$

$k^2 - 4k - 32 < 0$

$(k-8)(k+4) < 0$




$\therefore -4 < k < 8$

11 The roots of the equation $2x^2 - 8x + 50 = 0$ are $\alpha^2$ and $\beta^2$. Find
i) the value of $\alpha^2 + \beta^2$ and $\alpha^2\beta^2$.
ii) two different quadratic equations whose roots are $\alpha$ and $\beta$

i) $\alpha^2 + \beta^2 = 4$
$\alpha^2\beta^2 = 25$

ii) $\alpha\beta = \sqrt{25} = 5$
$(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$
$\alpha + \beta = \pm\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta}$
$\alpha + \beta = \pm\sqrt{4 + 2(5)}$
$= \pm\sqrt{14}$

$x^2 + \sqrt{14} + 5 = 0$
$x^2 - \sqrt{14} + 5 = 0$